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"ROBUST CONTROL OF SYSTEMS SUBJECT TO CONSTRAINTS"

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ROBUST CONTROL OF SYSTEMS SUBJECT TO CONSTRAINTS

Thesis by

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To Stephanie Jin-Yu Zheng, whose presence is solely missed

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Abstract

Most practical control problems are dominated by constraints. Although a rich theory has been developed for the robust control of linear systems, *very little* is known about the robust control of linear systems *with constraints.* Over the years various model-based algorithms (given a generic term Model Predictive Control) have been used in industry to control complex multivariable systems with operating constraints. The design and tuning of these controllers is difficult for two reasons:

1. Process models are always inaccurate which implies that the controllers must be *robust.*

2. Even in the simplest case where process models are linear, the overall systems are *nonlinear* because of the constraints.

Despite Model Predictive Control's considerable practical importance, there is *very little* theory to guide the design and tuning of these controllers for stability and robustness. It is the goal of this thesis to develop such a theory. Specifically, a general framework based on Model Predictive Control is developed to synthesize controllers for discrete-time linear systems subject to constraints with robust stability and performance guarantees.

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Chapter 1 Introduction

Process models are always inaccurate which implies the controllers designed must be *robust.* A rich theory [73] has been developed for the robust control of linear systems *without* constraints. The theory has been successfully applied to design robust controllers for a number of academic case studies such as high purity distillation columns [82]. However, industrial applications have not been as forthcoming. One main reason is that the current robustness theory does not take into account the fact that most practical control systems are *constrained.*

Most practical control problems are dominated by constraints. In the late 1970s and early 1980s, various model-based algorithms (given a generic term Model Predictive Control) (see, for example, [79, 20]) were developed by industrial researchers to control complex multivariable systems with operating constraints. The design and tuning of these controllers are difficult for two reasons: Firstly, process models are always inaccurate which implies that the controllers must be *robust.* Secondly, even in the simplest situation when process models are linear, the overall systems are *nonlinear* because of the constraints.

Despite Model Predictive Control's considerable practical importance and extensive use, there is *very little* theory to guide the design and tuning of these controllers for stability and robustness. It is the goal of this thesis to develop a general theory for designing controllers for linear discrete-time systems subject to *constraints* with robust stability and robust performance guarantees.

1.1 Motivation

Most practical control problems are dominated by constraints. There are generally two types of constraints—input constraints and output constraints. The input constraints are always present and are imposed by physical limitations of the actuators which cannot be exceeded under any circumstances. For example, valves can only be operated between fully open and fully closed, pumps and compressors have finite throughput capacity, and surge tanks can only hold a certain volume. Often, it is also desirable to keep specific outputs within certain limits for reasons related to plant operation, e.g. safety, material constraints, etc. For example, total impurities should be less than x for a distillation column, and reactors may have operating temperature and pressure limits. It may be, however, unavoidable to exceed the output constraints, at least temporarily, for example, when the system is subject to unexpected disturbances.

It may be argued that by overdesigning a controlled system the issue of physical limitations (input constraints) could be avoided. While this is true in principle, it is impractical due to the costs associated with the extra capacity built into the system which is never, or rarely, used. Indeed economic optimization of the system operating point typically *derives* the system to one or more constraints. Lee and Weekman [58] report

> "... in the petroleum industry the optimal operating point lies beyond the range of practical constraints. This probably occurs because of the savings incorporated into the design due to capital cost considerations. Thus a well designed plant *should* operate at a constraint, or it is really *overdesigned.*" (Emphasis added)

Lee and Weekman's comments were based on their experiences 20 years ago. With stiff competition and tight environmental regulations, today's processes are even more so than they were 20 years ago. Although Lee and Weekman's comments stem from the process industries, their economic considerations are valid in other disciplines as well. These include applications in aerospace, electrical, and mechanical engineering.

In addition to dealing with constraints at the controller design stage, it is important to recognize that process models are *always* inaccurate. Even for extremely detailed and involved first principles models, this will be true because assumptions and other simplifications made in deriving these models may *not* be satisfied and/or because parameters used may *not* be known *exactly*. Detailed models are typically difficult and costly to obtain. The costs associated with improved modeling must be balanced against the promise of improved control. Since there are diminishing returns in terms of control performance from improved modeling, *exact* modeling is not economically feasible.

As a result of model error (also called model uncertainty), the controller designed based on a model may not work as well, if at all, on the real plant. In fact, if model uncertainty is *not* taken into account properly, the performance on the real system can be arbitrarily bad (the overall system may even be unstable). The ultimate goal of designing a controller is for the controller to work on the real system, *not* on the model. Therefore, it is necessary that the controller should be designed to be insensitive to model uncertainty. We say that the controller is *robust* if small model uncertainty results in only small changes in performance. For linear systems *without* constraints, a rich theory has been developed to address the robustness issue (see, for example, the review article by Packard and Doyle [73] and the book by Dahleh and Diaz-Bobillo and references therein). However, *very little* is known for the robust control of linear systems *with constraints*. It is the aim of this thesis to develop such a theory for linear discrete-time systems with constraints.

1.2 Previous Work

Previous work on constraints and model uncertainty is summarized here.

1.2.1 Constraints

There are two popular approaches to design controllers for linear systems with constraints — Anti-Windup Bumpless Transfer (AWBT) and Model Predictive Control (MPC). There are, of course, many others (see, for example, [61, 91, 90, etc]), but we will not discuss them in this thesis. The AWBT design approach is based on the following two-step design paradigm: Firstly, a linear controller is designed by ignoring constraints. Because of the constraints, performance may suffer. In the next step, an anti-windup scheme is added to compensate for adverse effects of the constraints on closed loop performance. The AWBT design approach *rarely* deals with output constraints. The underlying principle of MPC is to determine some future control moves that optimize an open-loop performance objective over some horizon subject to input and output constraints. Although more than one control move is generally calculated at each sampling time, only the first control move is implemented. At the next sampling time, the output measurement is used to update the state estimate. The horizon is shifted forward by one sampling and the same calculations are repeated. This is why MPC is also referred to as Receding Horizon Control or Moving Horizon Control.

Anti-Windup Bumpless Transfer

Windup problems were originally encountered when using PI/PID controllers for controlling linear systems with control input nonlinearities. One of the earliest attempts to overcome windup in PID controllers was the work by Fertik and Ross [28]. It was recognized later, however, that integrator windup is only a special case of a more general problem. As pointed out by Doyle et al. [26], any controller with relatively slow or unstable modes will experience windup problems if there are actuator constraints. Windup is then interpreted as a mismatch between the controller output and the plant input when the control signal saturates. The "conditioning technique" as an AWBT scheme was originally formulated by Hanus et al. [40, 39] as an extension of the back calculation strategy of Fertik and Ross [28] to a general class of controllers. Åstrom et al. [1, 2] proposed that an observer be introduced into the system to estimate the states of the controller and hence restore consistency between the saturated control signal and the controller states. Walgama and Sternby [93] have very clearly exposed this inherent observer property in several anti-windup schemes. Campo and Morari [11] have derived the Hanus conditioned controller as a special case of the observer-based approach.

All these anti-windup schemes have been developed only for single-input single-output (SISO) systems. The extension to multi-input multi-output (MIMO) systems has not been attempted in its entirety. As pointed by Doyle et al. [26], for MIMO controllers, the saturation may cause a change in the plant input direction resulting in disastrous consequences. Through an example, Doyle et al. [26] showed that all of the existing anti-windup schemes failed to work on MIMO systems.

The stability analysis problem for SISO systems with input nonlinearity was extensively studied in the 1960s (see, for example, the book by Narendra and Taylor [72]). However, most stability results, e.g. circle conditions [81, 99] and off-axis criterion [12], were derived based on the standard conic sector bounded nonlinearity stability theory. It is well known that these results can be very conservative when applied to systems with input saturation constraints. Furthermore, the extension to MIMO systems *nonconservatively* was not straightforward. The issue of robustness has been largely ignored.

Recently Campo [9] and Kothare et al. [48] unified all existing AWBT schemes and developed a general framework for studying stability and robustness issues. The importance of this work lies in that model uncertainty can be taken into account systematically and powerful theory exists to analyze the closed loop system for stability and robustness. However, their analysis is also based on the standard conic sector nonlinear stability theory. Therefore, the results could be potentially conservative. Another drawback for all AWBT schemes is their inability to handle output constraints which may be present.

Model Predictive Control

In the late 1970s and early 1980s, various MPC algorithms (see, for example, [20, 79]) were developed in industry to control complex multivariable systems with input and output constraints. Some particular names include Model Predictive Heuristic Control (MPHC), Dynamic Matrix Control (DMC), Model Algorithm Control (MAC), Quadratic Dynamic Matrix Control (QDMC), and Identification and Command (IDCOM). MPC has been successfully implemented on process systems as diverse as distillation and oil fractionation [79, 41], fluid catalytic cracking [76, 36], hydrocracking [19, 46], and pulp and paper processing [62].

Because of the constraints, the overall MPC systems become nonlinear. Until recently when the infinite horizon MPC with guaranteed nominal stability was introduced by Rawlings and Muske [77], proving nominal stability for MPC systems represented a major obstacle [97]. An alternate but essentially equivalent approach is to enforce an end constraint [45], i.e. that the state at the end of a finite horizon must be zero (or more generally, within some region). (Some of the early work is due to Kwon and Pearson [53], but the ideas have seen a revival recently [15, 16, 60].) This approach is *identical* to setting the output horizon to infinity when the system is represented by a Finite Impulse Response (FIR) model and when the output horizon is chosen long enough for the system to settle.

Despite MPC's considerable practical importance and extensive use, there has been *very little* theory to guide the design and tuning of MPC controllers for *robustness*. Campo and Morari [10, 9] made the first rigorous attempt to extend the MPC concept to the control of *uncertain* linear systems and proposed a robust MPC algorithm. Unfortunately, it is well known (see, for example, [102]) that robust stability is not guaranteed with this algorithm. Zafiriou [96] used the contraction mapping principle to derive some necessary conditions and some sufficient conditions for robust stability. However, the conditions are both conservative and difficult to verify. Assuming lower and upper bounds on each impulse response coefficient, Genceli and Nikolaos [32] showed how to determine weights such that robust stability can be guaranteed for a set of FIR models. However, often weights do *not* exist even when robust stabilization is *possible* for a set of FIR models. Lee et al. [56] proposed a robust MPC algorithm that minimizes the expectation of a multi-step quadratic objective function for an input-output model with stochastic parameters. Of course, the concept of robust stability cannot be defined in this framework. For time-varying systems, Kothare et al. [49] proposed a robust MPC algorithm whose optimization problem for the state feedback case can be cast as a set of Linear Matrix Inequalities (LMIs) and showed that global asymptotic stability can be guaranteed if the optimization problem is feasible. This algorithm may be conservative when applied to linear time-invariant systems (see Chapter 7 for an example). Polak and Yang [75] proposed a receding horizon control strategy for linear continuous-time systems with input constraints and proved nominal stability of the closed loop system. Then they showed that robust stability is guaranteed *provided* that the perturbation is sufficiently small. The MPC concept has been extended to nonlinear systems. Discussing nonlinear MPC, however, is beyond the scope of this thesis. Interested readers are referred to the work by Mayne and Michalska [63, 64] and de Oliveira and Morari [23] for details.

1.2.2 Model Uncertainty

In stark contrast to the problem of constraints, a rich theory has been developed for the robust control of linear systems. Quantitative robustness analysis results were first articulated by Doyle and Stein [27] for unstructured model uncertainty, and by Doyle [25] for structured model uncertainty. General synthesis techniques have also been developed. For a recent description of these techniques, see the review article by Packard and Doyle [73]. For similar results obtained by using the l_1 approach, see the book by Dahleh and Diaz-Bobillo [21] and references therein.

The theory has substantially improved the ability of control system designers to develop multivariable designs for linear systems. It has not, however, been useful in designing AWBT compensation schemes or MPC controllers. This is because these systems include constraints which are not admitted by the theory.

1.3 Thesis Overview

In Chapter 2, we will give a brief tutorial review of the state-space formulation of MPC. Through an example, we show that under the still popular assumption of a finite output horizon it is difficult to provide stability guarantees that are general enough to be of practical value. By extending the output horizon to infinity or including an additional constraint called "end constraint," the stability question is reduced to the question of feasibility of the resulting optimization problem. The chapter finishes with some discussion on the feasibility of both input and output constraints.

It turns out that the output constraints may be infeasible for *stable* systems. As a result, the Infinite Horizon MPC with Mixed Constraints[1] (IHMPCMC) algorithm was introduced. In the next three chapters, we will investigate stability properties of the IHMPCMC algorithm for stable systems, systems with poles on the unit circle, and unstable systems (systems with poles outside the unit circle), respectively.

In Chapter 3, we show that global stability with the IHMPCMC algorithm is guaranteed for linear discrete-time stable systems with both state feedback and output feedback. The on-line optimization problem can be cast as a finite dimensional quadratic program even though the output constraints are specified over an infinite horizon. An example illustrates the main difference between the IHMPCMC algorithm and the Infinite Horizon MPC algorithm proposed by Rawlings and Muske [77].

Based on the growth rate of the set of states reachable with unit-energy inputs, we show in Chapter 4 that a discrete-time controllable linear system is globally controllable to the origin with *energy* bounded inputs[2] if and only if all its eigenvalues lie in the closed unit disk. These results imply that the IHMPCMC algorithm is semi-globally stabilizing for a sufficiently long input horizon if and only if the controlled system is stabilizable and all its eigenvalues lie in the closed unit disk. The disadvantage of this IHMPCMC algorithm is that the input horizon necessary for stabilization depends on the initial condition and can be arbitrarily large. As a result, we propose an implementable IHMPCMC algorithm. We show that with this algorithm a discrete-time linear system with n poles on the unit disk (with any multiplicity) can be globally stabilized if the input horizon is larger than n. For pure integrator systems, this condition is also necessary. Moreover, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input.

In Chapter 5, we analyze and characterize the domain of attraction for a linear

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[1] Mixed constraints refer to "hard" input constraints and "soft" output constraints.

[2] An energy bounded input refers to the following: Given any input $u(k) \in \Re^n$, $\sum_{i=1}^{\infty} u(i)^T u(i) < \infty$.

unstable discrete-time system with bounded controls. An algorithm is proposed to construct the domain of attraction. We show that the IHMPCMC algorithm (with a proper choice of the input horizon) generates a class of control laws that stabilize the system for all initial conditions in the domain of attraction.

The results from Chapters 3, 4, and 5 imply that the IHMPCMC algorithm, with the input horizon chosen properly, can globally stabilize any linear discrete-time system for which global stabilization is possible. If global stabilization is not possible (which is the case for unstable systems with constraints), the IHMPCMC algorithm stabilizes *any* initial condition for which a stabilizing control law exists.

In Chapter 6, we generalize the robust MPC algorithm proposed by Campo and Morari [10] for control of linear time-varying systems (represented by FIR models) with constraints. We show that with this scheme robust Bounded-Input Bounded-Output stability is guaranteed. Both necessary and sufficient conditions for global asymptotic robust stability are stated. Furthermore, we show that robust global asymptotic stability is preserved for a class of asymptotically constant disturbances entering at the plant output. Although these results hold for any uncertainty description expressed in the time-domain, there is a trade-off between the generality of the uncertainty description and the computational complexity of the resulting optimization problem. For a broad class of uncertainty descriptions, we show that the optimization problem can be cast as a linear program of moderate size.

In Chapter 7, we consider robust control of linear time-invariant systems with constraints. We propose a novel MPC algorithm which optimizes performance subject to stability constraints for linear systems with mixed constraints. In the nominal case, we show that global asymptotic stability is guaranteed for both state feedback and output feedback for linear time-invariant stable systems. Furthermore, global asymptotic stability is preserved for all asymptotically constant disturbances. The algorithm is then generalized to the robust case. We show that robust global asymptotic stability is guaranteed for a set of linear time-invariant stable systems. When the system is represented by an FIR model, we show that the optimization problem can be cast as a quadratic program of moderate size for a broad class of uncertainty descriptions.

Chapter 8 summarizes the contributions of this thesis work. In addition, suggestions for future research work are given. In Appendix A, a general anti-windup design which optimizes the error between the constrained output and the unconstrained output of the system, applicable to MIMO systems, is developed. The method generalizes the Model State Feedback for single-input multi-output systems proposed by Coulibaly et al. [17] and Hanus's conditioning technique [39, 40]. Furthermore, from our problem formulation, we can see what these methods do and why they do not work well on MIMO systems.

Chapter 2 Model Predictive Control

Summary

A tutorial review of the state-space formulation of Model Predictive Control is presented. The relations of Model Predictive Control to Internal Model Control and Linear Quadratic Gaussian control are briefly examined. We show through an example that under the still popular assumption of a finite output horizon it is difficult to provide stability guarantees that are general enough to be of practical value. By extending the output horizon to infinity or including an additional constraint called "end constraint," the stability question is reduced to the question of feasibility of the resulting optimization problem. The chapter finishes with some discussions on *global* feasibility of both input and output constraints.

2.1 Introduction

During the last two decades, various forms of Model Predictive Control (MPC) have become common in the process industries. Some particular names include Model Predictive Heuristic Control (MPHC), Dynamic Matrix Control (DMC), Model Algorithm Control (MAC), Quadratic Dynamic Matrix Control (QDMC), and Identification and Command (ID-COM). Many applications of MPC are reported in the literature and even more in sales publications. Some of them are mentioned in the review paper by Garcia et al. [31] and in the more recent summary article by Richalet [78]. MPC also enjoys widespread use in the Japanese process industries, as one can learn from the survey published by [95]. It is most significant that in a similar survey ten years prior [42], MPC can not even be found in the list of control techniques. MPC may be the most successful and widely accepted "advanced" control technique in process industry because

- MPC handles input and output constraints;
- MPC handles systems with the time delays;
- MPC is multivariable; and
- MPC is intuitive.

There is little doubt that most of the research on MPC started with the publication of the seminal papers by Cutler and Ramaker [20] from Shell and Richalet et al. [79]. This is not to suggest that they invented MPC, but they did convince a generation of control consultants, application engineers, managers, and researchers of the merits and the potential of this type of tool for industrial applications. Early joint work by Amoco and IBM [18, 51, 74] contains some of the essential features, but does not take into account process dynamics. There is also the theoretical work on "open-loop optimal feedback" with references going back to 1962 which is reviewed in the thesis by Gutman [37].

The various implementations of MPC preferred by the different vendors and users are identical in their main structure, but differ in details. These details are largely proprietary and are often critical for the success of the algorithm in an application. The general structure is shown in Figure 2.1. An observer utilizes knowledge of the plant input u and the output measurement y to arrive at a state estimate $\hat{x}$. Starting from the current state estimate $\hat{x}$, one can employ classic prediction algorithms to predict the behavior of the process output over some output horizon H_p when the manipulated input u is changed over some input horizon H_c (Figure 2.2).

At time step k, the task of the optimizer is to compute the present and future manipulated variable moves $\{u(k), \ldots, u(k+H_c)\}$ such that the predicted output follows the reference trajectory in a desirable manner. The optimizer takes into account constraints on the inputs and outputs which may be present. For linear process models, depending on the objective function, either a linear or a quadratic program results which is solved on-line in real-time at each time step. For commercial applications, various vendors have developed short-cut optimization procedures.

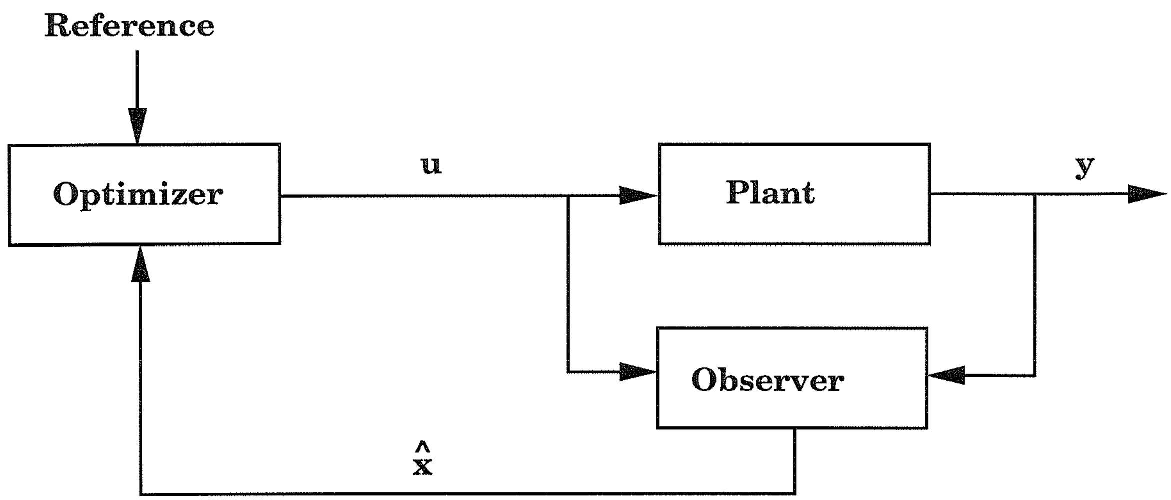

Figure 2.1: Structure inherent in all MPC schemes

Only $u(k)$, the first one of the sequence of optimal control moves is implemented on the real plant. At time step $k+1$, another output measurement $y(k+1)$ and another state estimate $\hat{x}(k+1)$ are obtained, the horizons are shifted forward by one step, and another optimization is carried out. This procedure results in a *moving horizon* or *receding horizon* strategy. A key feature of the technique is that the input and output horizons (H_c and H_p, respectively) are generally *finite*. Often the values chosen for H_c and H_p are different. Furthermore, in some of the algorithms, there is the option not to include the control error during the first few time steps in the objective function. The problem definition as presented allows one to treat with equal ease multivariable problems with an unequal number of inputs and outputs, non-minimum phase systems and systems subject to constraints.

The rest of the chapter is organized as follows. Section 2.2 gives a brief tutorial of the state-space formulation of MPC. For the input/output formulation of MPC, interested readers are referred to the book by Soeterboek [83] who provides an excellent exposition of the input/output formulation and assumptions. The relations of MPC to Internal Model Control and Linear Quadratic Gaussian control are briefly examined in Section 2.3. In Section 2.4, we show through an example that under the still popular assumption of a finite output horizon it is difficult to provide stability guar-

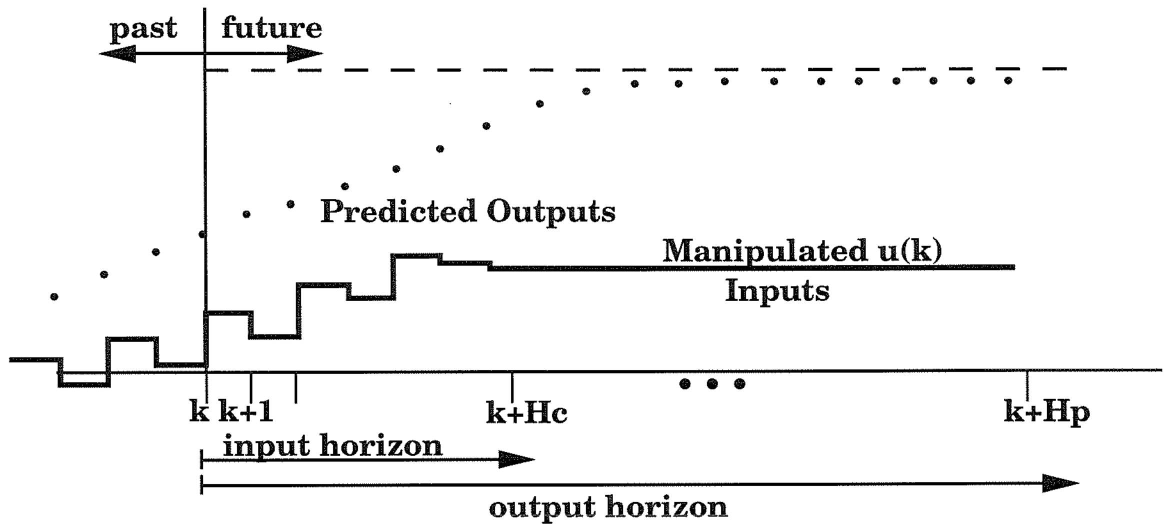

Figure 2.2: Definition of the optimization problem for MPC

antees that are general enough to be of practical value. By including an additional constraint called "end constraint" (Section 2.5) or extending the output horizon to infinity (Section 2.6), the stability question is reduced to the question of feasibility of the resulting optimization problem. We discuss global feasibility conditions for both input constraints and output constraints in Section 2.7. Section 2.8 concludes the chapter.

Notations and Assumptions The notation used in this chapter is fairly standard. $|\bullet|$ denotes the Euclidean norm, $|\bullet|_1$ the 1−norm, and $|\bullet|_\infty$ the ∞−norm. x^T denotes the transpose of x. For $x, y \in \Re^n, x \leq y$ if and only if $x_i \leq y_i, i = 1, \cdots, n$. We will assume throughout this chapter that the system to be controlled is linear time invariant discrete-time. For simplicity but without loss of generality, the disturbance and the noise are not included in the system. A good treatment of the disturbance and noise is given in [57]. Also we assume that we would like to keep the state at the origin rather than at some arbitrary reference state.

2.2 Problem Formulation

Assume that the system is described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{2.1}$$

where $x(k) \in \Re^{n_x}$ denotes the state at time step k, $u(k) \in \Re^{n_u}$ the manipulated variables (or the input), and $y(k) \in \Re^{n_y}$ the controlled variables (or the output). It is well known (see, for example, the paper by Lee et al. [57]) that the popular step response models used, for example, in Dynamic Matrix Control and other algorithms are just a special realization of a state-space model. Here we have not included the disturbance and noise for simplicity. The theory for output prediction is well developed (see, for example, [3] and [35]). It is summarized in the following:

$$x(k|k-1) = Ax(k-1|k-1) + Bu(k-1) \tag{2.2}$$

$$y(k|k-1) = Cx(k|k-1) \tag{2.3}$$

Correction based on measurements:

$$x(k|k) = x(k|k-1) + K(y(k) - y(k|k-1)) \tag{2.4}$$

Prediction:

$$x(k+1|k) = Ax(k|k) + Bu(k) \tag{2.5}$$

$$y(k+1|k) = Cx(k+1|k) \tag{2.6}$$

The filter gain K is determined from the solution of a Riccati equation. Prediction for more than one step ahead is obtained by applying the prediction equations recursively. Here $(\bullet)(k+i|k)$ denotes the variable at time step $k+i$ with information up to time k. Clearly, $x(k+i|k) = x(k+i) \ \forall \ i \leq 0$.

2.2.1 Objective Function

Various objective functions have been used. The most common one uses the 2−norm both spatially and temporally.

$$\begin{aligned}\Phi_k = & \sum_{i=1}^{H_p} x(k+i|k)^T \Gamma_x x(k+i|k) + \sum_{i=0}^{H_c} u(k+i|k)^T \Gamma_u u(k+i|k) \\ & + \sum_{i=0}^{H_c} \Delta u(k+i|k)^T \Gamma_{\Delta u} \Delta u(k+i|k)\end{aligned} \tag{2.7}$$

where

H_p is the output horizon

H_c is the input horizon

$\Delta u(k+i|k) = u(k+i|k) - u(k+i-1|k), \Delta u(k+i|k) = \Delta u(k+i) \ \forall\ i \leq 0$

Γ_x, Γ_u, and $\Gamma_{\Delta u}$ are positive definite (or semi-definite) weighting matrices

In general, one can even choose weighting matrices to be time varying, i.e. Γ_x, Γ_u, and $\Gamma_{\Delta u}$ may be functions of i. However for simplicity we assume them to be time-invariant here. Other popular objective functions are given as follows.

$1-1$ norm:

$$\Phi_k = \sum_{i=1}^{H_p} |\Gamma_x x(k+i|k)|_1 + \sum_{i=0}^{H_c} [|\Gamma_u u(k+i|k)|_1 + |\Gamma_{\Delta u} \Delta u(k+i|k)|_1]$$

$\infty - 1$ norm:

$$\Phi_k = \sum_{i=1}^{H_p} |\Gamma_x x(k+i|k)|_\infty + \sum_{i=0}^{H_c} [|\Gamma_u u(k+i|k)|_\infty + |\Gamma_{\Delta u} \Delta u(k+i|k)|_\infty]$$

$\infty - \infty$ norm:

$$\Phi_k = \max_{i=1,\cdots,H_p} |\Gamma_x x(k+i|k)|_\infty + \max_{i=0,\cdots,H_c} |\Gamma_u u(k+i|k)|_\infty + \max_{i=0,\cdots,H_c} |\Gamma_{\Delta u} \Delta u(k+i|k)|_\infty$$

$1-\infty$ norm:

$$\Phi_k = \max_{i=1,\cdots,H_p} |\Gamma_x x(k+i|k)|_1 + \max_{i=0,\cdots,H_c} |\Gamma_u u(k+i|k)|_1 + \max_{i=0,\cdots,H_c} |\Gamma_{\Delta u} \Delta u(k+i|k)|_1$$

A good description of advantages of each, especially the $\infty - \infty$ norm, as well as some other objective functions is given by Campo [9].

2.2.2 Constraints

There are generally two types of constraints—input constraints and output constraints. The input constraints can be described by imposing lower and upper bounds on the input.

$$u(k) \in \mathcal{U} = \left\{u : u^{min} \leq u \leq u^{max}\right\}, k \geq 0$$

Sometimes the rate of change of the input may be bounded, i.e.

$$|\Delta u(k)| \leq \Delta u^{max} \ \forall\ k$$

The output constraints can be described generally by

$$x(k) \in \mathcal{X} = \left\{x : [F_u \quad F_x] \begin{bmatrix} u \\ x \end{bmatrix} \leq f, u \in \mathcal{U}\right\}, k \geq 0$$

Clearly, to make any control problem meaningful, we must assume that $u = 0$ and $x = 0$ are an *interior* point of $\mathcal{U}$ and an *interior* point of $\mathcal{X}$, respectively, and that $\Delta u^{max} > 0$. As we shall see later, these constraints may be infeasible even for stable systems.

2.2.3 Control Design

The control actions are generated by *Controller MPC* which is defined as follows.

Definition 1 Controller MPC: *At time step k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c|k)} \Phi_k$$

$$\text{subject to} \quad \begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, 1, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, H_c + 1, \cdots, H_p \\ x(k+i|k) \in \mathcal{X} & i = 1, \cdots, H_p \end{cases} \tag{2.8}$$

For the objective function that uses the 2−norm both spatially and temporally, the optimization problem (2.8) can be cast as a quadratic program. For all others mentioned above, the optimization problem (2.8) can be cast as a linear program.

2.3 Relations to Other Methods

In this section, we discuss briefly how MPC *without* constraints is related to Internal Model Control and Linear Quadratic Gaussian control. Here we will assume that the 2−norm is used both spatially and temporally (i.e. Φ_k in Definition 1 is defined by (2.7)).

2.3.1 Internal Model Control

Without input and output constraints, the optimization problem (2.8) can be solved as a standard linear least squares problem. With the moving horizon assumption, a linear time invariant controller results. Garcia and Morari [30] have shown how to obtain the controller transfer function from the linear least squares solution.

Garcia and Morari [29] were the first to show that the structure, which is referred to as Internal Model Control (IMC), depicted in Figures 2.3, is inherent in all MPC schemes *without* constraints. Here P is the plant, $\tilde{P}$ a model of the plant, and Q_1 and Q_2 the controllers. It is well known [69] that the IMC structure and the classic feedback structure shown in Figure 2.4 are equivalent. However, the advantage of using the IMC structure is that closed loop stability is guaranteed if and only if Q_1 and Q_2 are stable when P is stable and $P = \tilde{P}$.

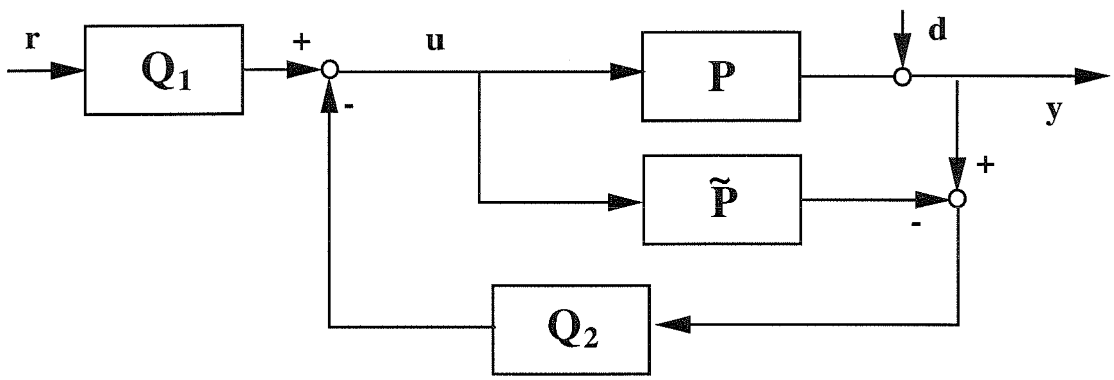

Figure 2.3: Internal Model Control structure

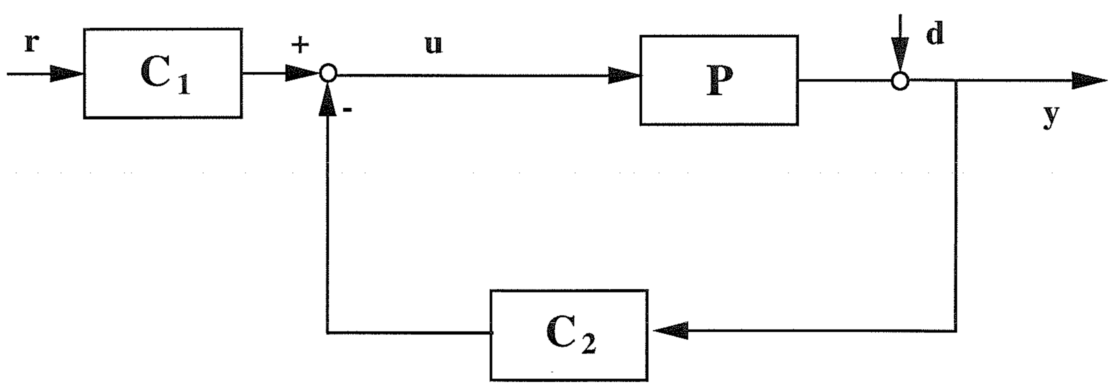

Figure 2.4: Classical feedback control structure

Much research has been done to relate various MPC tuning parameters to Q_1 and Q_2 and choose the tuning parameters properly so that Q_1 and Q_2 are stable. However, it is fair to say, after a decade of research, that such relationship, if it exists, is too complicated to be practical.[1] With $\Gamma_u = 0$, it can be easily shown that *Controller*

[1]It was falsely claimed in the survey paper [31] (Theorem 2 in the paper) that a sufficiently large

MPC provides integral control, i.e. no offset for step-like disturbances.

2.3.2 Linear Quadratic Gaussian Control

With $H_p = H_c = \infty$ and $\Gamma_{\Delta u} = 0$, and *without* constraints, the well studied infinite horizon Linear Quadratic Gaussian (LQG) optimal control problem results, which has been studied extensively for decades [52, 7, 54]. It has some nice properties, most importantly that the resulting controller is a constant gain acting either on the state, if available, or the state estimate, and that closed loop stability can be guaranteed under rather general conditions.

With H_p and H_c finite, some main differences between MPC and LQG are given as follows. Interested readers are referred to the paper by Garcia et al. [31] for more details.

- The MPC computation requires the solution of a linear least squares problem. LQG involves solving an algebraic Riccati equation.
- MPC has two more tuning parameters (H_c and H_p) than LQG.
- Most MPC algorithms used in the industry assume no measurement noise and step disturbance.

2.4 Finite Horizon MPC

Ever since MPC was first introduced in the late 70s and early 80s, much of the research has been done based on the assumption that both the input horizon and the output horizon are finite. Several reasons have been mentioned. Among them are

- *Simpler computation:* In certain situations, it may be simpler to use the MPC approach to find the controller gain matrix via a least squares problem, rather than by solving a Riccati equation which is necessary in the infinite horizon case.

weight on Δu would result in stable Q_1 and Q_2. See Section 2.4 for a counter example.

- *Constraints:* It is not immediately clear how a problem involving constraints on both inputs and outputs can be addressed in an infinite horizon setting.

- *More tuning flexibility:* The variable output horizon length (H_p) may offer another tuning parameter to achieve improved performance and robustness.

Unfortunately in retrospect there is little merit to these and other arguments in favor of a finite horizon approach.

- *Simpler computation:* With today's computer power at our disposal, the computational issue is largely irrelevant.

- *Constraints:* We can argue that the constrained case can be handled in an infinite horizon setting ($H_c = H_p = \infty$) as well. Let us assume for simplicity that we are regulating the state from some initial state x_0 to the origin and that the optimization problem is feasible, i.e. there exists a solution $u(k), u(k+1), \ldots$ which satisfies all the constraints and brings the state back to the origin. Clearly, the steady state solution $u^{ss} = 0$, $x^{ss} = 0$ is feasible and inside the constraint set. Thus, the problem is only constrained initially when the state is far from the origin and becomes unconstrained after sufficiently long time. This time can be estimated from some simple norm arguments. Therefore, we can solve the constrained problem over an infinite horizon by appropriately splicing together the solution for a constrained finite horizon and an unconstrained infinite horizon problem.

- *More tuning flexibility:* Tuning of control systems based on a finite horizon approach is often exceedingly difficult. The effect of the available parameters is often non-monotonic as demonstrated by Soeterboek [83]. For example, with $\Gamma_x = I$, $\Gamma_u = 0$ and $\Gamma_{\Delta u} = \gamma I$, increasing the input weight γ, which one would expect to suppress control action and stabilize the system, can actually destabilize a system. Upon further increase of the parameter, stable behavior is found. This is shown in Figure 2.5. This behavior is not observed with $H_p = \infty$ (Figure 2.6).

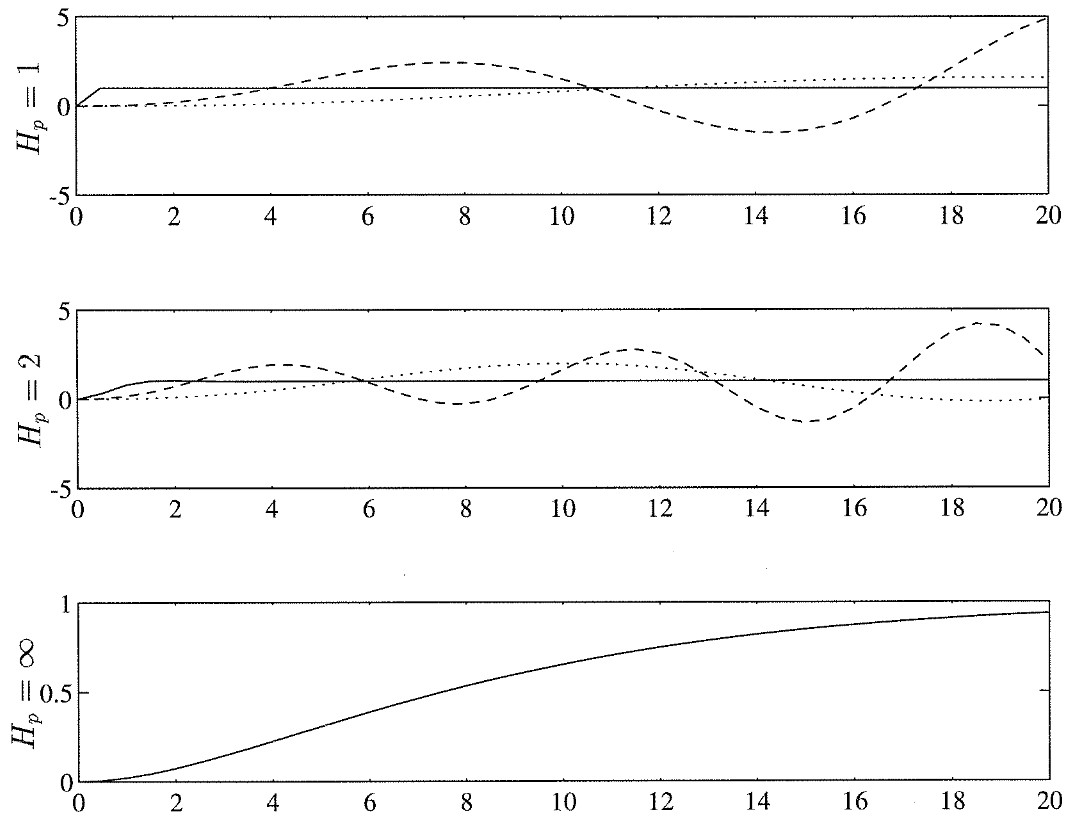

Figure 2.5: System $5/(4s+1)(5s+1); T_s = 0.5; H_c = 1$. For finite output horizons $H_p = 1$ or 2 the system behavior is "non-monotonic" as the input weight γ penalizing Δu is increased ($\gamma = 0$ solid; $\gamma = 0.1$ dash; $\gamma = 1$ dot)

As pointed out by Bitmead et al. [4], proving strong stability for the Finite Horizon MPC (FHMPC) formulation has been extremely unsuccessful. The stability results which have been obtained for the FHMPC formulation are all very weak (see, for example, the early results in [29, 14, 13].) They either are of an asymptotic nature, utilizing the well known results for the infinite horizon problem, or apply to very particular situations only (a specific class of systems, deadbeat control, etc.). In fact, we will now consider an example which illustrates that there does *not* exist a universal set of tuning parameters for the FHMPC formulation, within the input/output setting, that would guarantee stability for all systems.

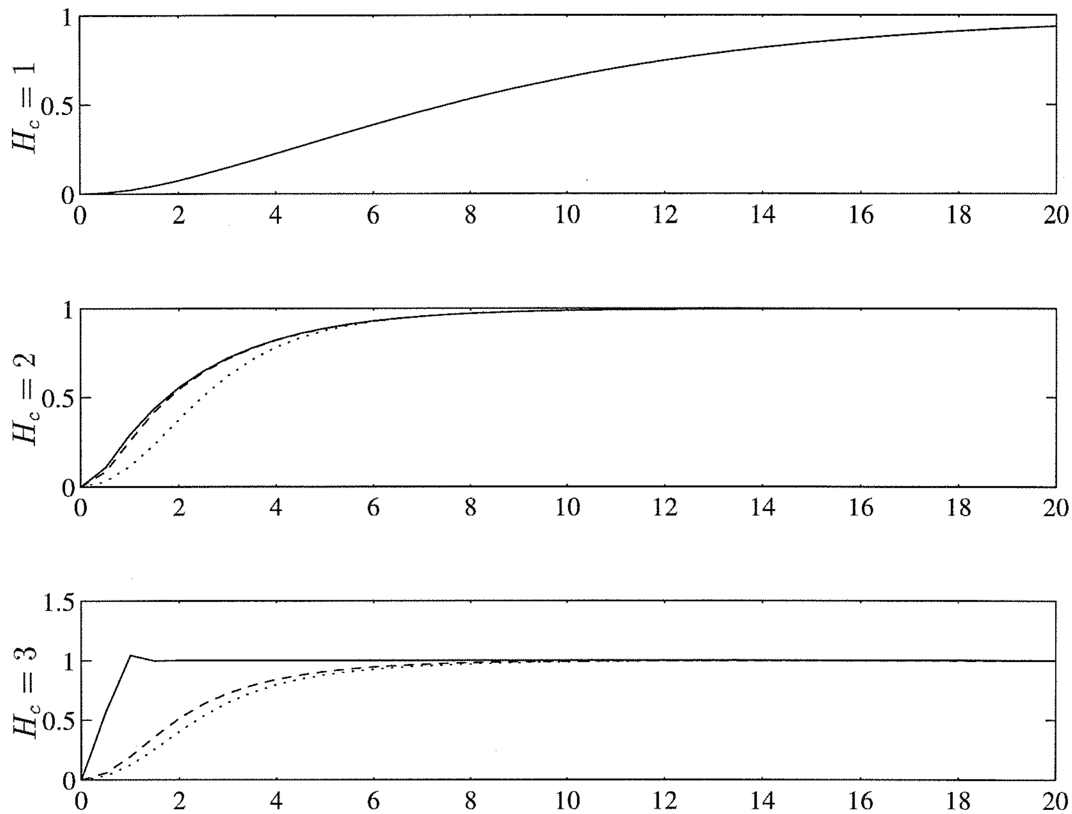

Figure 2.6: Same system as in Figure 2.5. $H_p = \infty$. For any input horizon H_c the system behavior is "monotonic" as the input weight ρ penalizing Δu is increased ($\gamma = 0$ solid; $\gamma = 0.1$ dash; $\gamma = 1$ dot)

Example 1 *Consider the following system.*

$$y(k) = -u(k-1) + u(k-2) + u(k - H_p - 1)$$

Suppose that there are no constraints and that there is no model/plant mismatch. The control action $u(k)$ *equals the first element* $u(k|k)$ *of the sequence* $\{\Delta u(k|k), \cdots, u(k+H_c-1|k)\}$ *which is the solution of the optimization problem*

$$\min_{\Delta u(k|k),\cdots,u(k+H_c|k)} \sum_{i=1}^{H_p} (r(k) - y(k+i|k))^2 + \Gamma_{\Delta u} \sum_{i=0}^{H_c-1} \Delta u(k+i|k)^2$$

The reason that no penalty on u *is used (i.e.* $\Gamma_u = 0$*) is to obtain integral control. What we want to show is the following: regardless of what* $\Gamma_u \geq 0$ *and* H_c *are, the* *closed loop system is always unstable. All we have to show is that the IMC controller Q_1 shown in Figure 2.3 is unstable. Using the formula given in [100], we have*

$$Q_1(q) = -\frac{1}{1 - q^{-1} - \frac{1}{\Gamma_{\Delta u}+1} q^{-H_p}}$$

Since $\frac{1}{Q_1(\infty)} = 1 > 0$ and $\frac{1}{Q_1(1)} = -\frac{1}{\Gamma_{\Delta u}+1} < 0 \ \forall \ 0 \leq \Gamma_{\Delta u} < \infty$, $\frac{1}{Q_1(q)}$ has roots outside the unit circle which implies that Q_1 is unstable. Thus the closed loop system is unstable for all $\Gamma_{\Delta u} \geq 0$ and H_c. Clearly the closed system would also be unstable if there are input and/or output constraints.

Notice that this example applies to the input/output formulation of the FHMPC. It may not apply to the state-space formulation of the FHMPC. The reason is that we may not be able to select the system order to be H_p as we have done here. However, this example does illustrate the problem with the FHMPC formulation and that it may be *necessary* to impose additional constraints (such as an end constraint) in order to guarantee stability.

We should point out that it may be possible to derive strong stability results if we make the output horizon H_p to be *dependent* on the system, i.e. given any system, closed loop stability may be guaranteed for a sufficiently large H_p. Of course, one simple way to get rid of this dependency is to choose $H_p = \infty$ [77].

2.5 Finite Horizon MPC with End Constraint

In order to prove general stability results for the FHMPC formulation, some additional constraints may have to be introduced. Several researchers [53, 45, etc] have proposed explicitly to include an additional constraint called "end constraint." The idea here is to force the state at the end of the output horizon to zero (or more generally within some region [65]), i.e. $x(k + H_p|k) = 0$. We refer to the resulting controller (or algorithm) as the Finite Horizon MPC with End Constraint (FHMPCEC) controller (or algorithm) which is defined below.

Definition 2 Controller FHMPCEC *At time step k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k)} \Phi_k$$

$$\text{subject to} \quad \begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, 1, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, H_c + 1, \cdots, H_p \\ x(k+H_p|k) = 0 & \\ x(k+i|k) \in \mathcal{X} & i = 1, \cdots, H_p \end{cases} \tag{2.9}$$

where Φ_k is defined by (2.7).

The idea of including the end constraint $x(k+H_p|k) = 0$ seems to be originated by Kwon and Pearson [53] for the unconstrained case although it was implicitly used in Kleiman's stabilizing controllers [47] (see Theorem 3 below). Keerthi and Gilbert [45] proved that closed loop stability can be guaranteed with *Controller FHMPCEC* in the presence of input and output constraints *provided* that the optimization problem (2.9) is feasible. We present the theorem below. We sketch the proof of the theorem because the ideas are simple and instructive.

Theorem 1 (State Feedback) *Consider the system described by (2.1). Assume that the state is measured and that there is no model/plant mismatch. Suppose that $\Gamma_x > 0, \Gamma_u > 0$, and $\Gamma_{\Delta u} \geq 0$. Then the closed loop system with* Controller FHMPCEC *is asymptotically stable to the origin if and only if the optimization problem (2.9) is feasible.*

Proof. Feasibility of the optimization problem (2.9) implies $J_k < \infty \ \forall \ k$. Since the optimal control sequence $\{u(k|k), \cdots, u(k+H_c-1|k)\}$ computed at time step k is feasible at time step $k+1$, we have

$$J_{k+1} \leq J_k - \left(x(k)^T \Gamma_x x(k) + u(k)^T \Gamma_u u(k) + \Delta u(k)^T \Gamma_{\Delta u} \Delta u(k)\right) \quad \forall\ k$$

It follows that

$$J_{k+1} + \sum_{i=0}^{k} \left(x(i)^T \Gamma_x x(i) + u(i)^T \Gamma_u u(i) + \Delta u(i)^T \Gamma_{\Delta u} \Delta u(i)\right) \leq J_0 < \infty$$

This together with $\Gamma_u, \Gamma_x > 0$ implies $u(k), x(k) \to 0$ asymptotically. □

Remark 1 *In order for the end constraint $x(k+H_p|k) = 0$ to be feasible, the system described by (2.1) must be* controllable.

Remark 2 *With state feedback, feasibility of the optimization problem (2.9) at sampling time* 0 *implies feasibility for all future sampling times. However, this may* not *be the case when the state has to be estimated and/or when there are disturbances.*

For the system described by (2.1), Kleiman provides a formula for stabilizing controllers when there are no constraints on inputs and outputs.

Theorem 2 (Kleiman 1974) *Consider the system described by (2.1). Assume that the system is controllable, that there are no constraints on the input and the output, that A is invertible, and that the state is measured. Then the closed loop system is stable with the following state feedback control law*

$$u(k) = -\Gamma_u^{-1} B^T (A^T)^{H_c-1} \left(\sum_{i=0}^{H_c-1} A^i B \Gamma_u^{-1} B^T (A^T)^i \right)^{-1} A^{H_c-1} A x(k) \qquad (2.10)$$

for all $H_c \geq n_x + 1$.

As it turns out, the idea of including an end constraint was also *implicitly assumed* here. Specifically, we can prove the following theorem.

Theorem 3 *Consider the system described by (2.1). Assume that the system (2.1) is controllable, that there are no input and output constraints, and that the state is* *measured. Then the feedback control law generated by* Controller FHMPCEC *with* $\Gamma_x = 0, \Gamma_{\Delta u} = 0$*, and* $H_p = H_c \geq n_x + 1$ *equals the feedback control law (2.10).*

Proof. With $\Gamma_x = 0$ and $\Gamma_{\Delta u} = 0$, the optimization problem (2.9) becomes

$$\min_{u(k|k),\cdots,u(k+H_c-1|k)} \sum_{i=0}^{H_c-1} u(k+i|k)^T \Gamma_u u(k+i|k) \quad \text{subject to } x(k+H_c|k) = 0$$

which is equivalent to

$$\min_{u(k|k),\cdots,u(k+H_c-1|k)} x(k+H_c|k)^T Q x(k+H_c|k) + \sum_{i=0}^{H_c-1} u(k+i|k)^T \Gamma_u u(k+i|k), Q = \infty I$$

After some algebra, the optimization problem becomes

$$\min_{U(k)} \left(A^{H_c}x(k) + \mathcal{C}_{H_c}U(k)\right)^T Q \left(A^{H_c}x(k) + \mathcal{C}_{H_c}U(k)\right) + U(k)^T \tilde{\Gamma}_u U(k)$$

where

$$\begin{aligned}
U(k) &= [u(k|k) \ \cdots \ u(k+H_c-1|k)]^T \\
\mathcal{C}_{H_c} &= \left[A^{H_c-1}B \ \cdots \ AB \ B\right] \\
\tilde{\Gamma}_u &= \begin{bmatrix} \Gamma_u & 0 & \cdots & 0 \\ & \vdots & & \\ 0 & \cdots & 0 & \Gamma_u \end{bmatrix}
\end{aligned}$$

The solution is given by

$$\begin{aligned}
U(k) &= -\left(\tilde{\Gamma}_u + \mathcal{C}_{H_c}^T Q \mathcal{C}_{H_c}\right)^{-1} \mathcal{C}_{H_c}^T Q A^{H_c} x(k) \\
&= -\left(\tilde{\Gamma}_u^{-1} - \tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T(\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T + Q^{-1})^{-1}\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\right) \mathcal{C}_{H_c}^T Q A^{H_c} x(k) \\
&= -\left(\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T Q - \tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T(\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T + Q^{-1})^{-1}\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T Q\right) A^{H_c} x(k) \\
&= -\left(\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T Q - \tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T \left(I + (\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T)^{-1}Q^{-1}\right)^{-1} Q\right) A^{H_c} x(k) \\
&= -\left(\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T Q - \tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T \left(I + (\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T)^{-1}Q^{-1}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
& +((\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T)^{-1}Q^{-1})^2 + \cdots \Big) Q\Big) A^{H_c}x(k) \\
= & -\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T \left((\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T)^{-1} - ((\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T)^{-2}Q^{-1}) - \cdots\right) A^{H_c}x(k) \\
= & -\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T(\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T)^{-1}A^{H_c}x(k)
\end{aligned}
$$

The second step uses the matrix inversion lemma while the last three steps follow from the fact that $Q^{-1} = 0$. Here we have assumed that $\mathcal{C}_{H_c}\tilde{\Gamma}_u^{-1}\mathcal{C}_{H_c}^T$ is invertible which is the case if $\mathcal{C}_{H_c}$ has full row rank which is implied by controllability of $\{A, B\}$ and $H_c \geq n_x + 1$ (see the discussion below). Since $u(k) = [I \;\; 0 \;\; \cdots \;\; 0]U(k)$, where I denotes the identity matrix, we have

$$
u(k) = -\Gamma_u^{-1}B^T(A^T)^{H_c-1}\left(\sum_{i=0}^{H_c-1} A^iB\Gamma_u^{-1}B^T(A^T)^i\right)^{-1} A^{H_c-1}Ax(k)
$$

which is the same as (2.10). □

From this theorem, we can clearly see what Kleiman's controllers do. Also from our proof, several assumptions in Theorem 2 can either be relaxed or ignored. A necessary and sufficient condition on H_c for (2.10) to be stabilizing is that H_c is such that $\mathcal{C}_{H_c}$ has full row rank. So the assumption that $H_c \geq n_x + 1$ can be replaced by that H_c is such that $\mathcal{C}_{H_c}$ has full row rank. Furthermore, the assumption that A is invertible can be dropped. The controllability assumption is needed since without it the state *cannot* be made identically zero, i.e. $x(k + H_c|k) = 0$ is infeasible.

In the proof, we interpreted the end constraint as an infinite weight on the state at the end of output horizon, while the weights on the states for the rest of horizon are finite. Several stability results [4] have been proved by assuming Γ_x to be time-varying. Recently, the idea of including an end constraint to enforce stability has seen a revival. Interested readers are referred to [15, 16, 60, etc] for more recent developments.

2.6 Infinite Horizon MPC

As we discussed in Section 2.2.4, it may be true that, given any system, the closed loop stability can be guaranteed for a sufficiently large H_p. Of course, this H_p depends on the system. One way to remove this dependency is to have $H_p = \infty$. Indeed this is what has been suggested by Rawlings and Muske [77]. The resulting algorithm is referred to as the Infinite Horizon MPC (IHMPC) algorithm which is defined as follows.

Definition 3 Controller IHMPC *At time step k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k)} \Phi_k$$

$$\text{subject to} \quad \begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, 1, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, H_c + 1, \cdots, H_p \\ x(k+i|k) \in \mathcal{X} & i = 1, \cdots, H_p \end{cases} \tag{2.11}$$

where Φ_k is defined by (2.7) and $H_p = \infty$.

Rawlings and Muske showed that closed loop stability can be guaranteed with *Controller IHMPC* if the optimization problem (2.11) is feasible. We sketch the proof of the theorem here since the ideas are simple and instructive.

Theorem 4 (State Feedback) *Consider the system described by (2.1). Assume that the state is measured and that there is no model/plant mismatch. Suppose that $\Gamma_x > 0, \Gamma_u > 0$, and $\Gamma_{\Delta u} \geq 0$. Then the closed loop system with* Controller IHMPC *is asymptotically stable if and only if the optimization problem (2.8) is feasible.*

Proof. Feasibility of the optimization problem implies $J_k < \infty \; \forall \; k$. Since the optimal control sequence $\{u(k|k), \cdots, u(k+H_c-1|k)\}$ computed at time k is feasible at time

$k+1$, we have

$$J_{k+1} \leq J_k - \left(x(k)^T\Gamma_x x(k) + u(k)^T\Gamma_u u(k) + \Delta u(k)^T\Gamma_{\Delta u}\Delta u(k)\right), \ \forall\ k$$

which yields

$$J_{k+1} + \sum_{i=0}^{k}\left(x(i)^T\Gamma_x x(i) + u(i)^T\Gamma_u u(i) + \Delta u(i)^T\Gamma_{\Delta u}\Delta u(i)\right) \leq J_0 < \infty$$

This together with $\Gamma_u, \Gamma_x > 0$ implies $u(k), x(k) \rightarrow 0$ asymptotically. □

Remark 3 *For the optimization problem (2.11) to be feasible, the system needs only to be* stabilizable. *On the other hand, as we remarked earlier, the system must be* controllable *for the optimization problem (2.9) (which is defined by* Controller FHMPCEC*) to be feasible. These approaches are* identical *when the system can be represented by a Finite Impulse Response (FIR) model and when the output horizon H_p in* Controller FHMPCEC *has been chosen long enough for the system to settle (i.e. $H_p \geq H_c + N_{FIR}$ where N_{FIR} is the order of the FIR model including the delays).*

While the work by Rawlings and Muske is exemplary in its clarity, it must be mentioned that other authors (see, for example, [63, 50]) have suggested independently to prove stability via the Lyapunov function J_k.

The implementation of the IHMPC algorithm (i.e. *Controller IHMPC*) is discussed in [71]. The basic idea is to break the objective function into two parts as follows:

$$\begin{aligned}\Phi_k &= x(k+H_c|k)^T\bar{\Gamma}_x x(k+H_c|k) + \sum_{i=0}^{H_c} x(k+i|k)^T\Gamma_x x(k+i|k) \\ &\quad + \sum_{i=0}^{H_c}\left(u(k+i|k)^T\Gamma_u u(k+i|k) + \Delta u(k+i|k)^T\Gamma_{\Delta u}\Delta u(k+i|k)\right)\end{aligned}$$

where $\bar{\Gamma}_x$ can be determined by solving a Lyapunov function. This effectively replaces the infinite output horizon with a finite horizon.

2.7 Feasibility of the Constraints

Both the FHMPCED and IHMPC algorithms have converted the question of closed loop stability into the question of feasibility of the resulting optimization problems. Feasibility of an optimization problem means that the objective function is bounded and that all constraints are satisfied. It is well known (see, for example, [84]) that a linear discrete-time system is *globally* stabilizable with input constraints if and only if the system is stabilizable and has all its eigenvalues inside the closed unit disk. Thus, the input constraints may be *infeasible* for unstable systems for some initial conditions. In general, input constraints are imposed by physical limitations of the system. They cannot be violated under any circumstances. Therefore, for unstable systems, we can only determine the region of initial conditions for which stabilization is *possible*.

On the other hand, output constraints can be *infeasible* even for stable systems. We illustrate this by considering the following example.

$$\begin{aligned} x(k+1) &= \begin{bmatrix} .655 & -0.1673 \\ .1673 & 0.9825 \end{bmatrix} x(k) + \begin{bmatrix} .1673 \\ 0.0175 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} -2 & 1 \end{bmatrix} x(k) \end{aligned}$$

The system is stable and has a zero outside the unit circle (inverse response behavior). Suppose that we would like to keep the output within ± 1 and that there are no input constraints. Then, *regardless* of how we choose the tuning parameters, output constraints are infeasible for the initial condition $x(0) = [1.5 \;\; 1.5]^T$. Furthermore, this implies that there does *not* exist a stabilizing controller that would satisfy the output constraints. Since this type of system is very common in process control applications, it is essential to have methods that deal with infeasible output constraints effectively. Two approaches have been proposed: Rawlings and Muske [77] suggested to ignore the infeasible output constraints and they showed that stability is preserved in this case. The other approach [80] is to relax the output constraints as follows

$$x(k) \in \mathcal{X}_\epsilon = \left\{ x : [F_u \quad F_x] \begin{bmatrix} u \\ x \end{bmatrix} \leq f + \epsilon, u \in \mathcal{U} \right\}, k \geq 0, \epsilon \geq 0$$

and penalize the extent of violation by adding a penalty term (e.g. $\epsilon^T \Gamma_\epsilon \epsilon$ with $\Gamma_\epsilon \geq 0$) to the objective function. This results in the Infinite Horizon MPC with Mixed Constraints (IHMPCMC) algorithm which is defined as follows.

Definition 4 Controller IHMPCMC: *At time step k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{\epsilon(k), u(k|k), \cdots, u(k+H_c-1|k)} \quad \Phi_k + \epsilon(k)^T \Gamma_\epsilon \epsilon(k)$$

$$\text{subject to} \quad \begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, \cdots, H_c - 1 \\ \Delta u(k+i|k) = 0 & i = H_c, \cdots, \infty \\ x(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 1, \cdots, \infty \end{cases} \tag{2.12}$$

where Φ_k is defined by (2.7) and $\Gamma_\epsilon \geq 0$ is diagonal.

In the next three chapters, we will investigate stability properties of *Controller IHMPCMC* for stable systems, systems with poles on the unit circle, and unstable systems (i.e. systems with poles outside the unit circle), respectively.

2.8 Conclusions

In this chapter, we have given a brief introduction to the state-space formulation of MPC. We refer interested readers to the books by Soeterboek [83] and Morari et al. [68] for details. We showed through an example why it is difficult to obtain general stability results for the finite horizon MPC formulation. This is entirely consistent with the fact that over the last two decades proving strong stability results for the finite horizon MPC formulation has been extremely unsuccessful [4]. In order to obtain strong stability results, it is time to revise the problem formulation. In fact, this is exactly what has been initiated by several research groups independently during the last couple of years and a wealth of existing results have appeared. Two of them that are discussed in some details in this chapter are the FHMPCEC algorithm and the IHMPC algorithm.

Both the FHMPCEC and IHMPC algorithms convert the question of stability into the question of feasibility of the resulting optimization problems. Unfortunately, the output constraints may be infeasible even for stable systems. As a result, the IHMPCMC algorithm was introduced. In the next three chapters, we will investigate stability properties of the IHMPCMC algorithm for stable systems, systems with poles on the unit circle, and unstable systems (i.e. systems with poles outside the unit circle), respectively.

Chapter 3 Infinite Horizon MPC with Mixed Constraints—Stable Systems

Summary

We show that with the Infinite Horizon Model Predictive Control with Mixed Constraints algorithm *global* asymptotic stability is guaranteed for linear discrete-time stable systems with both state feedback and output feedback. The on-line optimization problem defining the algorithm can be cast as a finite dimensional quadratic program even though the output constraints are specified over an infinite horizon.

3.1 Introduction

Many practical control problems are dominated by constraints. There are generally two types of constraints—input constraints and output constraints. The input constraints are always present and are imposed by physical limitations of the actuators which *cannot* be violated under any circumstances. For this reason, we refer to input constraints as "hard" constraints. Often, it is also desirable to keep specific outputs within certain limits for reasons related to plant operation, e.g. safety, material constraints, etc. It is usually unavoidable to exceed the output constraints, at least temporarily, for example, when the system is subjected to unexpected disturbances. Thus, output constraints are referred to as "soft" constraints.

In Chapter 2, we presented several results that show the equivalence between closed loop stability and feasibility of the respective optimization problem. Specifically, both with the Infinite Horizon MPC [77] and Finite Horizon MPC with End Constraint [45] algorithms, the closed loop system is asymptotically stable if and only if their resulting optimization problems are feasible. Unfortunately, as we showed through an example in Chapter 2, output constraints may be infeasible even for *stable* systems.

Several methods have been proposed to deal with infeasible output constraints. Rawlings and Muske [77] proposed to remove the infeasible output constraints during the *initial portion* of the infinite horizon to make the optimization problem feasible. However, this may result in undesirable performance: the violation of the output constraints during this initial portion of the infinite horizon can be very large in order to satisfy the constraints during the rest. Thus, large constraint violations may be experienced, when the computed control actions are implemented.

An alternative way to handle the feasibility problem is to relax the infeasible output constraints for the *entire* horizon and to penalize the extent of the violation. This technique is referred to as "constraint softening" [80]. The problem is that global stability may not be guaranteed. Zafiriou and Chiou [98] have derived some conditions for stability for single-input single-output systems with the finite horizon MPC formulation. However, these conditions are generally conservative and difficult to check. Softening the output constraints with the infinite horizon MPC formulation results in the Infinite Horizon MPC with Mixed Constraints (IHMPCMC) algorithm which was introduced in Chapter 2.

In this chapter, along with the next two chapters, we will investigate stability properties of the IHMPCMC algorithm for stable systems (this chapter), systems with poles on the unit circle (Chapter 4), and unstable systems (Chapter 5). This chapter is organized as follows. Sections 3.2 and 3.3 deal with state feedback and output feedback, respectively. Specifically, we show that global asymptotic stability is guaranteed in both cases. In addition, we show in Section 3.3 that the optimization problem can be cast as a finite dimensional quadratic program even though the output constraints are specified over the infinite horizon. An example is presented in Section 3.4. Section 3.5 concludes the chapter.

Notations The notation used in this chapter is fairly standard. $|\bullet|$ denotes the Euclidean norm, $|\bullet|_1$ the $1-$norm, and $|\bullet|_\infty$ the $\infty-$norm. x^T denotes the transpose of x. For $x, y \in \Re^n, x \leq y$ if and only if $x_i \leq y_i, i = 1, \cdots, n$.

3.2 State Feedback

Consider the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{3.1}$$

where $x(k) \in \Re^{n_x}, u(k) \in \Re^{n_u}$ and $y(k) \in \Re^{n_y}$.

The input is assumed to belong to the set $\mathcal{U}$ which is defined as follows.

$$u(k) \in \mathcal{U} \triangleq \{u : 0 > u^{min} \leq u \leq u^{max} > 0\} \tag{3.2}$$

The soft output constraints are defined as follows:

$$\mathcal{X}_\epsilon \triangleq \left\{ x : [F_x \quad F_u] \begin{bmatrix} x \\ u \end{bmatrix} \leq f + \epsilon, \epsilon \geq 0, u \in \mathcal{U} \right\} \tag{3.3}$$

The objective function is defined as follows.

$$\begin{aligned} \Phi_k = \sum_{i=1}^{\infty} x(k+i|k)^T \Gamma_x x(k+i|k) \quad &+ \sum_{i=0}^{H_c} \Big[u(k+i|k)^T \Gamma_u u(k+i|k) + \\ &\Delta u(k+i|k)^T \Gamma_{\Delta u} \Delta u(k+i|k) \Big] \end{aligned} \tag{3.4}$$

where $\Gamma_x > 0, \Gamma_u > 0, \Gamma_{\Delta u} \geq 0, \Delta u(k+i|k) = u(k+i|k) - u(k+i-1|k)$, and H_c is finite. Γ_x, Γ_u, and $\Gamma_{\Delta u}$ are symmetric. $(\cdot)(k+i|k)$ denotes the variable $(\cdot)$ at sampling time $k+i$ predicted at sampling time k and $(\cdot)(k) = (\cdot)(k|k)$.

The control actions are generated by *State Feedback Controller IHMPCMC* which is defined as follows.

Definition 5 State Feedback Controller IHMPCMC: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence* $\{u(k|k), u(k+$

$1|k), \cdots, u(k+H_c-1|k)\}$ *which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \quad \Phi_k + \epsilon(k)^T \Gamma_\epsilon \epsilon(k)$$

$$\text{subject to} \quad \begin{cases} |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, \cdots, H_c \\ u(k+i|k) \in \mathcal{U} & i = 0, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, \cdots, \infty \\ x(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0, 1, \cdots, \infty \\ \epsilon(k) \geq 0 & \end{cases} \tag{3.5}$$

where $\Gamma_\epsilon > 0$ *is diagonal and* Φ_k *is defined by (3.4).*

The output constraints are *softened* by the slack variables $\epsilon(k)$. They can be violated temporarily, if necessary. In the long term, the penalty term $\epsilon(k)^T \Gamma_\epsilon \epsilon(k)$ in the objective function will drive the slack variables to zero. The optimization problem (3.5) can be cast as a quadratic program.

The control problem is to bring the state to the origin. To make it well posed, the feasible region for

$$\begin{cases} |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c \\ u(k+i|k) \in \mathcal{U} & i = 0, 1, \cdots, H_c - 1 \end{cases}$$

must contain $u(k+i|k) = 0, i = 0, 1, \cdots, H_c - 1$, as an *interior* point. The feasible region for

$$\begin{cases} x(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0, 1, \cdots, \infty \\ \epsilon(k) = 0 & \end{cases}$$

contains $x(k+i|k) = 0, i = 0, 1, \cdots, \infty$, as an *interior* point. Note that this implies $f > 0$. Then we have the following theorem which extends the results in [77] for $\epsilon(k) = 0 \ \forall \ k \geq 0$.

Theorem 5 *The closed-loop system with* State Feedback Controller IHMPCMC *is globally asymptotically stable if and only if the optimization problem (3.5) is feasible for all* $x(0) \in \Re^{n_x}$.

Proof. If the optimization problem (3.5) is not feasible, the controller is not defined. Feasibility of the optimization problem implies that J_1 is finite. At sampling time $k+1$, let

$$\begin{cases} u^*(k+i|k+1) = u(k+i|k) \qquad i = 1, 2, \cdots, H_c \\ \epsilon^*(k+1) = \epsilon(k) \end{cases}$$

Thus, (u^*, ϵ^*) is a feasible solution but may not be optimal. We have

$$J_{k+1} \leq J_k - x(k+1)^T \Gamma_x x(k+1) - u(k)^T \Gamma_u u(k) - \Delta u(k)^T \Gamma_{\Delta u} \Delta u(k)$$

which yields

$$J_{k+1} + \sum_{i=1}^{k} \left[x(i+1)^T \Gamma_x x(i+1) + u(i)^T \Gamma_u u(i) + \Delta u(k)^T \Gamma_{\Delta u} \Delta u(k) \right] \leq J_1 < \infty$$

Note that we replaced $x(k+1|k)$ with $x(k+1)$ since $x(k+1) = x(k+1|k)$. This together with $\Gamma_x, \Gamma_u > 0$ implies that $x(k) \to 0$ and $u(k) \to 0$ as $k \to \infty$. □

Remark 4 *Theorem 5 also holds if* $\Gamma_u \geq 0$ *provided that at steady state* $x = 0$ *if and only if* $u = 0$ *which is equivalent to that* $(I - A)^{-1}B$ *has full column rank. Also if* $(I - A)^{-1}B$ *has full column rank, Theorem 5 holds with* $u(k+i|k) = 0, i = H_c, \cdots, \infty$ *in (3.5) replaced by* $\Delta u(k+i|k) = 0, i = H_c, \cdots, \infty$.

Remark 5 *If* $\Gamma_\epsilon = \infty$, *then the output constraints become hard and the optimization problem (3.5) may not be feasible.*

The following theorem states that for $\Gamma_\epsilon < \infty$ feasibility of the optimization problem (3.5) is guaranteed for stable systems.

Theorem 6 *If* A *is stable,* i.e. *all eigenvalues of* A *are strictly inside the unit circle, then the optimization problem (3.5) is feasible* $\forall\ H_c \geq 1,\ \Gamma_\epsilon < \infty$, *and* $x(0) \in \Re^{n_x}$.

Proof. All we have to do is to prove the feasibility of the optimization problem (3.5) at the first sampling time. We will prove this theorem by construction. Since A is stable, $x(k)$ is bounded $\forall\ k \geq 0$ for any initial condition. Then

$$\begin{cases} u^*(i|1) = 0 & i = 1, 2, \cdots, H_c \\ \epsilon^*(1) = \max\limits_{i \geq 1} |F_x x(i|1)|_\infty < \infty & \end{cases}$$

satisfies all the constraints and results in $J_1 < \infty$. Thus it is a feasible solution. □

Remark 6 *Theorems 5 and 6 hold as well if other norms for softening the output constraints are used.*

3.3 Output Feedback

In the previous section, we assumed that the state is measured. Since the closed loop system may be nonlinear because of the constraints, we cannot apply the Separation Principle to prove global stability for the output feedback case. It is well known that, in general, a nonlinear closed loop system with the state estimated via an exponentially converging observer can be unstable even though it is stable with state feedback. Although it is trivial to show *local* asymptotic stability here, proving *global* asymptotic stability is nontrivial. We will show in this section that *global* asymptotic stability of the closed loop system generated by *State Feedback Controller IHMPCMC* and an exponentially converging observer is guaranteed for stable systems.

Denote the state (output) at sampling time $k+i$ estimated at sampling time k by $\hat{x}(k+i|k)$ $(\hat{y}(k+i|k))$. The state is estimated as follows.

$$\begin{aligned} \hat{x}(k|k) &= A\hat{x}(k-1|k-1) + Bu(k-1) + K(y(k) - \hat{y}(k|k-1)) \\ \hat{x}(k+i|k) &= A\hat{x}(k+i-1|k) + Bu(k+i-1) \qquad i \geq 1 \end{aligned} \tag{3.6}$$

where K is the observer gain. Define *Output Feedback Controller IHMPCMC* as follows.

Definition 6 Output Feedback Controller IHMPCMC: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$\hat{J}_k = \min_{\epsilon(k), u(k|k), \cdots, u(k+H_c-1|k)} \quad \hat{\Phi}_k + \epsilon(k)^T \Gamma_\epsilon \epsilon(k)$$

$$\text{subject to} \quad \begin{cases} |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, \cdots, H_c \\ u(k+i|k) \in \mathcal{U} & i = 0, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, \cdots, \infty \\ x(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0, 1, \cdots, \infty \\ \epsilon(k) \geq 0 & \end{cases} \tag{3.7}$$

where $\Gamma_\epsilon > 0$ diagonal, $\hat{x}(\cdot|\cdot)$ estimated via Equation (3.6), and

$$\begin{aligned} \hat{\Phi}_k = \sum_{i=1}^{\infty} \hat{x}(k+i|k)^T \Gamma_x \hat{x}(k+i|k) \quad &+ \sum_{i=0}^{H_c} \Big[u(k+i|k)^T \Gamma_u u(k+i|k) \\ &+ \Delta u(k+i|k)^T \Gamma_{\Delta u} \Delta u(k+i|k) \Big] \end{aligned} \tag{3.8}$$

Combining Equations (3.6) and (3.1) yields

$$e(k+1) = (I - KC)Ae(k) \tag{3.9}$$

where $e(k) = x(k) - \hat{x}(k|k)$. Thus Equation (3.6) can be written as

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + KCAe(k-1) \\ \hat{x}(k+i|k) &= A\hat{x}(k+i-1|k) + Bu(k+i-1) \qquad i \geq 1 \end{aligned} \tag{3.10}$$

which yields

$$\begin{aligned} \xi(k|k) &= KCAe(k-1) \\ \xi(k+i|k) &= A\xi(k+i-1|k) \qquad i \geq 1 \end{aligned} \tag{3.11}$$

where $\xi(k+i|k) = \hat{x}(k+i|k) - \hat{x}(k+i|k-1)$.

Remark 7 *The overall system with* Output Feedback Controller IHMPCMC *can be expressed as follows.*

$$\begin{cases} x(k+1) & = g(x(k), e(k)) \\ e(k+1) & = (I - LC)Ae(k) \end{cases} \tag{3.12}$$

where $x(k+1) = g(x(k), 0)$ represents the closed loop system with state feedback and is globally asymptotically stable for stable systems. To prove global asymptotic stability for (3.12) is a special case of an actively studied problem (see, for example, [88],[85]) that considers a more general set of equations.

$$\begin{cases} \dot{x} & = g_1(x, e) \\ \dot{e} & = g_2(e) \end{cases} \tag{3.13}$$

where both $\dot{x} = g_1(x, 0)$ and $\dot{e} = g_2(e)$ are globally asymptotically stable.

Before we state the result on global asymptotic stability of the closed loop system with *Output Feedback Controller IHMPCMC*, let us first prove the following lemma.

Lemma 1 *Assume that A and $(I - KC)A$ are stable, i.e. all the eigenvalues are strictly inside the unit circle. Define*

$$\eta(k) = \max_{i \geq k} |F_x A^{i-k} e(i)|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \Re^{n_F}$$

Then

$$
\begin{aligned}
\sum_{k=1}^{\infty}\sqrt{\sum_{i=0}^{\infty}\xi(k+i|k)^T\Gamma_x\xi(k+i|k)} &< \infty \\
\sum_{k=1}^{\infty}\sqrt{\eta(k)^T\Gamma_\epsilon\eta(k)} &< \infty \\
\sum_{k=1}^{\infty}\left(\sqrt{\sum_{i=0}^{\infty}\xi(k+i|k)^T\Gamma_x\xi(k+i|k)}+\sqrt{\eta(k)^T\Gamma_\epsilon\eta(k)}\right)^2 &< \infty
\end{aligned}
\tag{3.14}
$$

where $0 \le \Gamma_x, \Gamma_\epsilon < \infty$ *and* n_F *is the number of columns of* $[F_x \quad F_u]$.

Proof. From Equations (3.9) and (3.11), we have

$$|e(k)|_2 \le c_1 k^{\alpha_1 - 1}\rho_1^k |e(0)|_2$$

and

$$|\xi(k+i|k)|_2 \le c_2 i^{\alpha_2-1}\rho_2^i|\xi(k|k)|_2 \le c_3 i^{\alpha_2-1}k^{\alpha_1-1}\rho_2^i\rho_1^k|e(0)|_2$$

where $\rho_1 = \lambda_{max}((I-KC)A)$ and $\rho_2 = \lambda_{max}(A)$; c_1, c_2, and c_3 are constant; α_1 and α_2 are the multiplicities associated with the largest eigenvalues[1] of $(I-KC)A$ and A, respectively. Here $\lambda_{max}(A)$ denotes the spectral radius of A. Stability of A and $(I-KC)A$ implies that $\rho_1, \rho_2 < 1$. Thus,

$$\sum_{k=1}^{\infty}\sqrt{\sum_{i=0}^{\infty}\xi(k+i|k)^T\Gamma_x\xi(k+i|k)} \le c_1c_2|e(0)|_2\bar{\sigma}(\Gamma_x^{\frac{1}{2}})\sqrt{\sum_{i=0}^{\infty}i^{2(\alpha_2-1)}\rho_2^{2i}}\sum_{k=1}^{\infty}k^{\alpha_1-1}\rho_1^k < \infty$$

The other two expressions can be proved similarly. □

Remark 8 *If A is unstable or has poles on the unit circle, Lemma 1 clearly does not hold.*

The following theorem states that global asymptotic stability with output feedback can be guaranteed for stable systems.

Theorem 7 *Assume that A and $(I-KC)A$ are stable,* i.e. *all eigenvalues of A and*

[1] The largest eigenvalue is defined to be the eigenvalue with the largest absolute value.

$(I-KC)A$ are strictly inside the unit circle. Then the overall system with Output Feedback Controller IHMPCMC *is globally asymptotically stable.*

Proof. Denote the weighted 2−norm $\sqrt{x^T R x}$ by $|x|_R$. Let

$$\begin{cases} u^*(k+i|k+1) = u(k+i|k) \qquad i = 1, 2, \cdots, m \\ \epsilon^*(k+1) = \epsilon(k) + \eta(k) \end{cases}$$

where $\eta(k)$ is as defined in Lemma 1. Thus, (u^*, ϵ^*) is a feasible solution but may not be optimal. Define

$$U = \sum_{i=1}^{H_c} \left[|u(k+i|k)|^2_{\Gamma_u} + |\Delta u(k+i|k)|^2_{\Gamma_{\Delta u}} \right]$$

$$V(k) = |\hat{x}(k+1|k)|^2_{\Gamma_x} + |u(k)|^2_{\Gamma_u} + |\Delta u(k)|^2_{\Gamma_{\Delta u}}$$

We have

$$\begin{aligned} \hat{J}_{k+1} &\leq \sum_{i=2}^{\infty} |\hat{x}(k+i|k+1)|^2_{\Gamma_x} + U + |\epsilon^*(k+1)|^2_{\Gamma_\epsilon} \\ &= \sum_{i=2}^{\infty} |\hat{x}(k+i|k) + \xi(k+i|k+1)|^2_{\Gamma_x} + U + |\epsilon(k) + \eta(k)|^2_{\Gamma_\epsilon} \\ &\leq \left(\sqrt{\sum_{i=2}^{\infty} |\hat{x}(k+i|k)|^2_{\Gamma_x} + U + |\epsilon(k)|^2_{\Gamma_\epsilon}} + \sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x} + |\eta(k)|_{\Gamma_\epsilon}} \right)^2 \\ &= \left(\sqrt{\hat{J}_k - V(k)} + \sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x} + |\eta(k)|_{\Gamma_\epsilon}} \right)^2 \end{aligned}$$

Taking square root on both sides yields

$$\begin{aligned} \sqrt{\hat{J}_{k+1}} &\leq \sqrt{\hat{J}_k - V(k)} + \sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x} + |\eta(k)|_{\Gamma_\epsilon}} \\ &\leq \sqrt{\hat{J}_k} + \sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x} + |\eta(k)|_{\Gamma_\epsilon}} \end{aligned}$$

which in turn yields

$$\sqrt{\hat{J}_{k+1}} \leq \sqrt{\hat{J}_1} + \sum_{j=1}^{k} \left[\sqrt{\sum_{i=2}^{\infty} |\xi(j+i|j+1)|^2_{\Gamma_x}} + |\eta(j)|_{\Gamma_\epsilon} \right]$$

By Lemma 1, the second term on the right-hand-side is bounded for all k. Therefore, we have

$$\hat{J}_k \leq J^{max} < \infty \qquad \forall \quad k > 0$$

From before, we have

$$\begin{aligned}
\hat{J}_{k+1} &\leq \left(\sqrt{\hat{J}_k - V(k)} + \sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right)^2 \\
&= \hat{J}_k - V(k) + \left(\sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right)^2 \\
&\quad +2\sqrt{\hat{J}_k - V(k)} \left(\sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right) \\
&\leq \hat{J}_k - V(k) + \left(\sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right)^2 \\
&\quad +2\sqrt{J^{max}} \left(\sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right)
\end{aligned}$$

which yields

$$\begin{aligned}
\hat{J}_{k+1} + \sum_{i=1}^{k} V(i) \leq \hat{J}_1 + \sum_{i=1}^{k} \Bigg[& \left(\sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right)^2 \\
& +2\sqrt{J^{max}} \left(\sqrt{\sum_{i=2}^{\infty} |\xi(k+i|k+1)|^2_{\Gamma_x}} + |\eta(k)|_{\Gamma_\epsilon} \right) \Bigg]
\end{aligned}$$

By Lemma 1 and boundness of J^{max}, the second term is bounded for all k. Thus,

$$\hat{J}_{k+1} + \sum_{i=1}^{k} V(i) = \hat{J}_{k+1} + \sum_{i=1}^{k} \left[|\hat{x}(i+1|i)|^2_{\Gamma_x} + |u(i)|^2_{\Gamma_u} + |\Delta u(i)|^2_{\Gamma_{\Delta u}} \right] < \infty$$

Following a similar argument as in the proof of Theorem 6, we can therefore conclude that $x(k) \to 0$ and $u(k) \to 0$ as $k \to \infty$. $\square$

The following theorem shows that the output constraints over the *infinite* horizon can be replaced by the output constraints over an *finite* horizon. A similar result was derived by Rawlings and Muske [77].

Theorem 8 *Assume that A is stable. Given any $\hat{x}(k|k)$ and $\epsilon(k) \geq 0$, there exists a finite N such that*

$$\hat{x}(k+i|k) \in \mathcal{X}_{\epsilon(k)} \quad \forall\, i \geq N$$

Proof. We need only prove this theorem for $\epsilon(k) = 0$: since $\epsilon(k) \geq 0\ \forall\, k$, $\hat{x}(k+i|k) \in \mathcal{X}\ \forall\, i \geq N$ implies $\hat{x}(k+i|k) \in \mathcal{X}_{\epsilon(k)}\ \forall\, i \geq N$. Suppose that $N > H_c$. Then we only have to show that $F_x\hat{x}(k+i|k) \leq f\ \forall\, i \geq N$. WLOG, assume that A is nonsingular.[2] Consider a zero input, *i.e.* $u(k+i|k) = 0, i = 0, \cdots, H_c - 1$, and denote the value of the objective function for this input sequence by $\hat{J}_k^*$. Then,

$$\hat{J}_k \leq \hat{J}_k^* = \hat{x}(k|k)^T \sum_{i=1}^{\infty} (A^T)^i \Gamma_x A^i \hat{x}(k|k) \equiv \hat{x}(k|k)^T \Pi \hat{x}(k|k)$$

where Π is positive definite and bounded since A is nonsingular and stable. Also we have

$$\begin{aligned}
\hat{J}_k &= \sum_{i=1}^{\infty} \hat{x}(k+i|k)^T \Gamma_x \hat{x}(k+i|k) + \epsilon(k)^T \Gamma_\epsilon \epsilon(k) \\
&\quad + \sum_{i=0}^{H_c} \left[u(k+i|k)^T \Gamma_u u(k+i|k) + \Delta u(k+i|k)^T \Gamma_{\Delta u} \Delta u(k+i|k) \right] \\
&\geq \sum_{i=H_c}^{\infty} \hat{x}(k+i|k)^T \Gamma_x \hat{x}(k+i|k) \\
&= \hat{x}(k+H_c|k)^T \Pi \hat{x}(k+H_c|k)
\end{aligned}$$

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[2] If A is singular, we can write $A = T^{-1} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} T$ where $\Sigma_1 > 0$ and Σ_2 is nilpotent. Define $\tilde{x}(k) = Tx(k)$ and $\begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix}$. Then, after a finite number of sampling times, $\tilde{x}_2$ becomes identically zero since Σ_2 is nilpotent. Thus it suffices to consider the reduced system with $\tilde{x}_1$ as its states.

Combining these two inequalities, we obtain

$$\hat{x}(k+H_c|k)^T\Pi\hat{x}(k+H_c|k) \le \hat{x}(k|k)^T\Pi\hat{x}(k|k)$$

which yields

$$|\hat{x}(k+H_c|k)|_2 \le \kappa(\Pi)|\hat{x}(k|k)|_2$$

where $\kappa(\Pi) < \infty$ denotes the condition number of Π. Finally,

$$\begin{aligned}
|F_x\hat{x}(k+H_c+N|k)|_\infty &= |F_xA^N\hat{x}(k+H_c|k)|_\infty \\
&\le |F_xA^N\hat{x}(k+H_c|k)|_2 \\
&\le \bar{\sigma}(F_x)\bar{\sigma}(A^N)|\hat{x}(k+H_c|k)|_2 \\
&\le \bar{\sigma}(F_x)\bar{\sigma}(A^N)\kappa(\Pi)|\hat{x}(k|k)|_2
\end{aligned}$$

where $\bar{\sigma}(F_x)$ denotes the largest singular value of F_x. If N is such that

$$\bar{\sigma}(F_x)\bar{\sigma}(A^{N+i})\kappa(\Pi)|\hat{x}(k|k)|_2 \le \min_j(f_j) \quad \forall\, i \ge 0$$

then

$$F_x\hat{x}(k+i|k) \le f \quad \forall\, i \ge N+H_c$$

$f > 0$ and stability of A imply that a finite N exists. □

3.4 Example

Consider the system

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 0.655 & -0.1673 \\ 0.1673 & 0.9825 \end{bmatrix} x(k) + \begin{bmatrix} 0.1637 \\ 0.0175 \end{bmatrix} u(k) \\
y(k) &= [-2 \;\; 1]x(k)
\end{aligned} \tag{3.15}$$

which is obtained from the continuous-time transfer function $\frac{-2s+1}{(s+1)^2}$ with a sampling time of 0.2. The initial condition is $x(0) = [1.5 \quad 1.5]^T$. The output is constrained between ± 1. Since the system exhibits inverse response behavior, hard output constraints can cause stability problems [96]. To use the approach proposed in [77], the output constraint at the first sampling time must be ignored to make the optimization problem feasible. We can also use the approach presented in this chapter and soften the output constraints over the infinite horizon. The following parameter values are used:

$$H_c = 5, \Gamma_x = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}, \Gamma_u = 0.1I, \Gamma_{\Delta u} = 0, \Gamma_\epsilon = I$$

where I is the identity matrix. Using the arguments leading to Theorem 8 one can show that the output constraints will be satisfied automatically after 35 time steps. Thus, the output constraints need only be enforced over a finite horizon of length 35. The responses for the two approaches are depicted in Figure 3.1. A very large overshoot is observed for the controller designed via the approach proposed in [77] but the output comes within the constraints faster.

Figure 3.2 shows responses with output feedback. The initial state estimate is $\hat{x}(0) = [0 \quad 0]^T$ and the observer gain is $K = [0.1 \quad 1]^T$.

3.5 Conclusions

We have analyzed stability properties of the IHMPCMC algorithm for linear time-invariant discrete-time stable systems. We showed that global asymptotic stability can be guaranteed for both state feedback and output feedback cases. The on-line optimization problem can be cast as a *finite* dimensional quadratic program. In the next two chapters, we will investigate stability properties of the IHMPCMC algorithm for systems with poles on the unit circle and unstable systems.

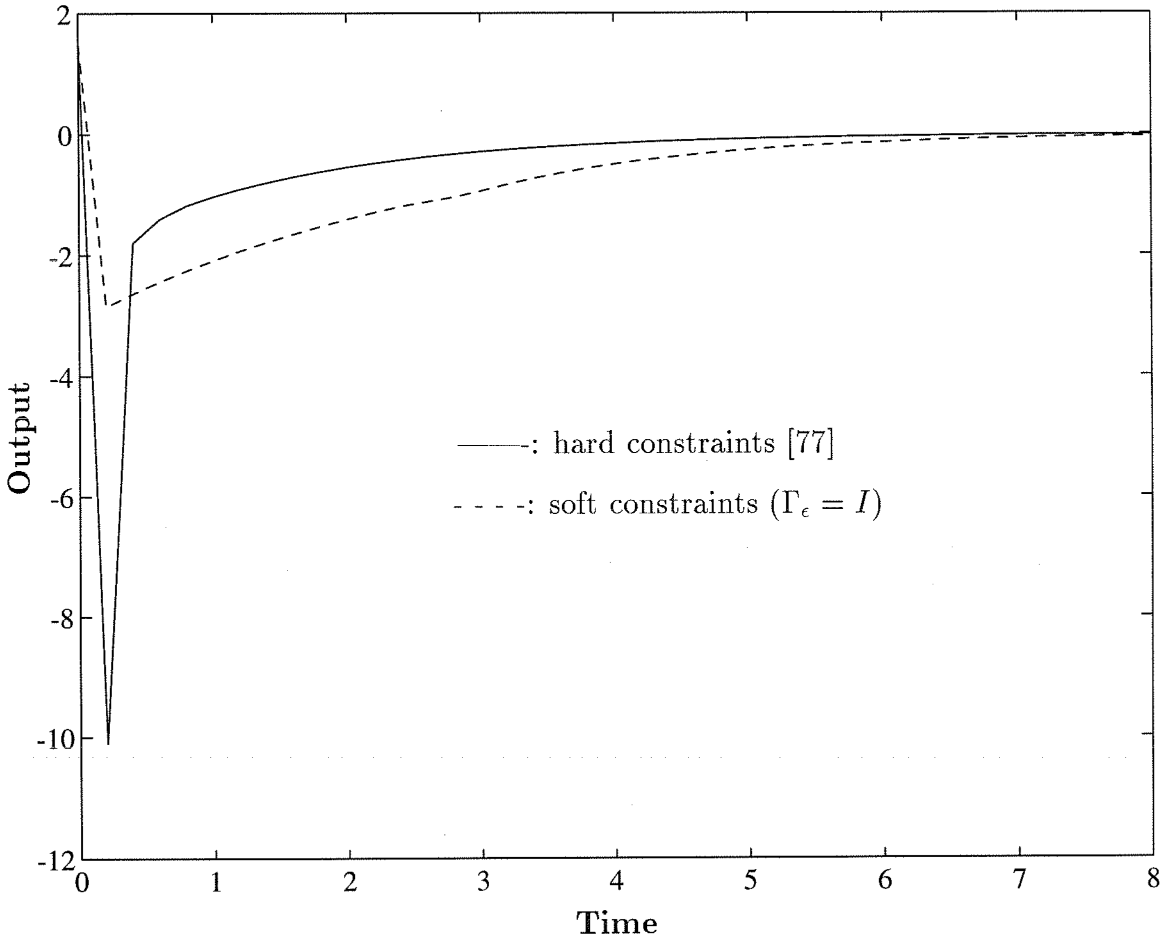

Figure 3.1: Comparison of responses for the two approaches

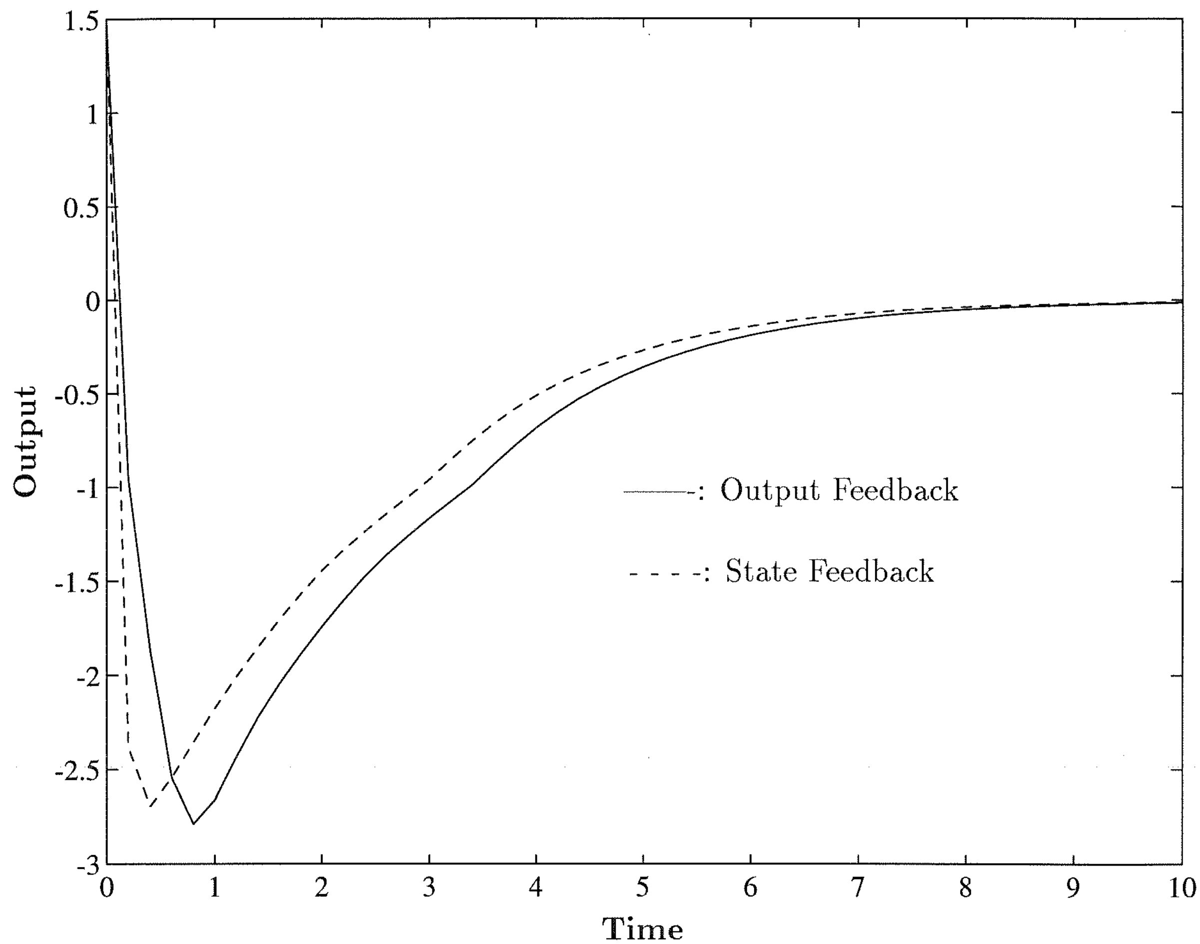

Figure 3.2: Output feedback responses

Chapter 4 Infinite Horizon MPC with Mixed Constraints—Systems with Poles on the Unit Circle

Summary

Based on the growth rate of the set of states reachable with unit-energy inputs, we show that a discrete-time controllable linear system is globally controllable to the origin with energy-bounded inputs if and only if all its eigenvalues lie in the closed unit disk. These results imply that the Infinite Horizon Model Predictive Control with Mixed Constraints algorithm is semi-globally stabilizing for a sufficiently long input horizon if and only if the controlled system is stabilizable and all its eigenvalues lie in the closed unit disk.

The disadvantage of the Infinite Horizon Model Predictive Control with Mixed Constraints algorithm is that the input horizon necessary for stabilization depends on the initial condition and can be arbitrarily large. As a result, we propose an implementable Infinite Horizon Model Predictive Control with Mixed Constraints algorithm. We show that with this algorithm a discrete-time linear system with n poles on the unit circle (with any multiplicity) can be globally stabilized if the input horizon is larger than n. For pure integrator systems, this condition is also necessary. Moreover, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input.

4.1 Introduction

It is well known [84] that a linear time-invariant discrete-time system is globally stabilizable with bounded inputs if and only if the system is stabilizable and all its eigenvalues are inside the closed unit disk. The problem of constructing both globally stabilizing and semi-globally stabilizing controllers for linear discrete-time systems with poles on the closed unit disk has been extensively studied over the last few years. Various approaches, e.g. optimal control [63, 92, 75], smooth nonlinear control [90, 86, 87, 91], and semi-global stabilization [61], have been employed to construct stabilizing controllers for such systems. In this chapter, we use Model Predictive Control (MPC) to study this problem.

Keerthi and Gilbert, and Rawlings and Muske [77] showed that, respectively, the Finite Horizon MPC with End Constraint algorithm and the Infinite Horizon MPC algorithm can globally stabilize linear discrete-time systems if and only the optimization problems defining these algorithms are feasible for all initial conditions (see also Chapter 2). We showed in the previous chapter that, with the Infinite Horizon MPC with Mixed Constraints (IHMPCMC) algorithm, the optimization problem is guaranteed to be feasible for *stable* systems. The question now is: is the optimization problem defining the IHMPCMC algorithm always feasible for systems with poles on the unit circle?

It was shown in [92] that for stabilizable systems with poles in the closed unit disk, given any initial condition, the optimization problem is *always* feasible, *provided* that the input horizon (H_c) is sufficiently long. Conversely [92, 86], for systems with poles outside the unit disk, there *always* exist initial conditions for which the optimization problem is infeasible. In this chapter, we prove the same result under stronger assumptions on the input: Based on the growth rate of the set of states reachable with *unit-energy* inputs (i.e. $\sum_{i=0}^{\infty} u(i)^T u(i) \leq 1$), we show that a discrete-time controllable linear system is globally controllable to the origin with unit-energy inputs if and only if all its eigenvalues lie in the closed unit disk. Then we show that the IHMPCMC algorithm is semi-globally stabilizing for a sufficiently long input horizon. However, the input horizon needed for feasibility of the optimization problem depends on the initial condition; it is generally difficult to determine *a priori* and can be arbitrarily large. Furthermore, in practice an unmeasured disturbance could still cause the optimization problem to become infeasible and an even larger number of control moves may have to be chosen. Therefore, this strategy is *not* easily implementable.

As a result, we propose an implementable IHMPCMC algorithm. We show that with this scheme a discrete-time linear system with n poles on the unit circle (with any multiplicity) can be globally stabilized if the input horizon is larger than n (i.e. $H_c \geq n+1$). For the specific case of a chain of n integrators, this condition is also necessary. Furthermore, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input.

This chapter is organized as follows: In Section 4.2, we show that the singular values of the ellipsoidal set of states reachable in N steps with unit energy inputs for a discrete-time $n-$integrator system grow as $\{O(N^{2n-1}), O(N^{2n-3}), \cdots, O(N)\}$. This implies that a discrete-time controllable linear system is globally controllable to the origin if and only if all its eigenvalues lie in the closed unit disk. In Section 4.3, we show that the IHMPCMC algorithm is semi-globally stabilizing. In Section 4.4, we propose an implementable IHMPCMC algorithm and show that this scheme is globally stabilizing if the input horizon is larger than the number of poles on the unit circle (with any multiplicity). Two examples are presented in Section 4.5. Section 4.6 concludes the chapter. For notational simplicity, the results in Section 4.4 are proved for single-input single-output (SISO) systems. We discuss the extension of the results to multi-input multi-output (MIMO) systems.

Notations The notation used in this chapter is fairly standard. $|\bullet|$ denotes the Euclidean norm, $|\bullet|_1$ the $1-$norm, and $|\bullet|_\infty$ the $\infty-$norm. x^T denotes the transpose of x. $\sqrt{x^T R x} = |x|_R$. For $x, y \in \Re^n, x \leq y$ if and only if $x_i \leq y_i, i = 1, \cdots, n$. $O(N)$ means in the order of N.

4.2 Constrained Stabilizability of Linear Discrete-Time Systems

In this section, we give the necessary and sufficient condition for the *existence* of a control law to globally stabilize a discrete-time linear system subject to *energy bounded* inputs (i.e. $\sum_{i=0}^{\infty} u(i)^T u(i) \leq 1$). The result is stronger than the stabilization result proved by Sontag [84] which applies to bounded inputs (i.e. $|u(i)|_\infty \leq 1 \ \forall\, i \geq 0$).

4.2.1 Reachable Set for a Multiple-Integrator System

Consider the discrete-time integrator chain

$$x(k+1) = A_n^J x(k) + e_{\text{last}} u(k) \tag{4.1}$$

where A_n^J is a Jordan block of size n with eigenvalue 1:

$$A_n^J = \begin{bmatrix} 1 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

and e_{last} is the last Euclidean basis vector, that is, $e_{\text{last}} = [0 \ \cdots \ 0 \ 1]^T$. The size of e_{last} will be determined from context (of course, here $e_{\text{last}} \in \Re^n$).

The set of states reachable with unit-energy inputs in N steps for system (4.1) is

$$\mathcal{R}_N \triangleq \left\{ z \;\middle|\; x(0) = 0,\ x(N) = z,\ x(\cdot) \text{ satisfies (4.1) and } \sum_{k=0}^{N-1} u(k)^2 \leq 1 \right\} \tag{4.2}$$

Of course, $\mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots \subseteq \mathcal{R}_N$. Moreover, it is well known [8] that $\mathcal{R}_N$ is the ellipsoid

$$\left\{ z \;\middle|\; z^T W_{n,N}^{-1} z \leq 1 \right\}$$

where $W_{n,N} = \sum_{k=0}^{N-1} (A_n^J)^k e_{\text{last}} e_{\text{last}}^T ((A_n^J)^T)^k$. We will refer to $W_{n,N}$ as the $N-$step reachability Gramian of the pair (A_n^J, e_{last}). We denote the ith singular value of $W_{n,N}$ by $\sigma_i(W_{n,N})$, and state the following result:

Theorem 9 *The n singular values of $W_{n,N}$, $\{\sigma_1(W_{n,N}), \sigma_2(W_{n,N}), \dots, \sigma_n(W_{n,N})\}$, are*

$$\{O(N^{2n-1}), O(N^{2n-3}), \dots, O(N)\}$$

in N. Moreover, the corresponding singular vectors of $W_{n,N}$ converge to the standard Euclidean basis of $\Re^n$ $\{[1\ \ 0\cdots 0],[0\ \ 1\cdots 0],\ldots,[0\ \ 0\cdots 1]\}$.

Proof. We first note that

$$(A_n^J)^k e_{\text{last}} = \left[\begin{pmatrix} k \\ n-1 \end{pmatrix} \begin{pmatrix} k \\ n-2 \end{pmatrix} \cdots \begin{pmatrix} k \\ n-n \end{pmatrix}\right]^T$$

where

$$\begin{pmatrix} m \\ n \end{pmatrix} \triangleq \begin{cases} \dfrac{m!}{n!(m-n)!} & \text{if } m \geq n \\ 0 & \text{otherwise} \end{cases}$$

Therefore, W_N equals

$$\sum_{k=0}^{N-1} \begin{bmatrix} \begin{pmatrix} k \\ n-1 \end{pmatrix}\begin{pmatrix} k \\ n-1 \end{pmatrix} & \begin{pmatrix} k \\ n-1 \end{pmatrix}\begin{pmatrix} k \\ n-2 \end{pmatrix} & \cdots & \begin{pmatrix} k \\ n-1 \end{pmatrix}\begin{pmatrix} k \\ n-n \end{pmatrix} \\ \begin{pmatrix} k \\ n-2 \end{pmatrix}\begin{pmatrix} k \\ n-1 \end{pmatrix} & \begin{pmatrix} k \\ n-2 \end{pmatrix}\begin{pmatrix} k \\ n-2 \end{pmatrix} & \cdots & \begin{pmatrix} k \\ n-2 \end{pmatrix}\begin{pmatrix} k \\ n-n \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} k \\ n-n \end{pmatrix}\begin{pmatrix} k \\ n-1 \end{pmatrix} & \begin{pmatrix} k \\ n-n \end{pmatrix}\begin{pmatrix} k \\ n-2 \end{pmatrix} & \cdots & \begin{pmatrix} k \\ n-n \end{pmatrix}\begin{pmatrix} k \\ n-n \end{pmatrix} \end{bmatrix}$$

In the sequel, given matrices A and B that depend on k, we will say "$A(k) \approx B(k)$ for large k" to mean that $\lim_{k\to\infty} A_{ij}(k)/B_{ij}(k) = 1$. Then, since

$$\begin{pmatrix} k \\ n-j \end{pmatrix} = \frac{k(k-1)\cdots(k-n+j+1)}{(n-j)!}$$

we have

$$\begin{pmatrix} k \\ n-j \end{pmatrix} \approx \frac{k^{(n-j)}}{(n-j)!}$$

for large k and

$$\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-i \end{pmatrix} \begin{pmatrix} k \\ n-j \end{pmatrix} \approx \frac{N^{(2n-i-j+1)}}{(n-i)!\,(n-j)!\,(2n-i-j+1)} \tag{4.3}$$

for large N.

Therefore, we conclude that for large N, W_N is

$$\begin{bmatrix} O(N^{2n-1}) & O(N^{2n-2}) & \dots & O(N^{2n-n}) \\ O(N^{2n-2}) & O(N^{2n-3}) & \dots & O(N^{2n-n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ O(N^{2n-n}) & O(N^{2n-n-1}) & \dots & O(N^{1}) \end{bmatrix} \tag{4.4}$$

Intuition suggests that this fact means that the largest singular value $\sigma_1(W_{n,N})$ grows as $O(N^{2n-1})$ and the corresponding left and right singular vectors tend to $e_1 = [1 \quad 0 \cdots 0]$, that the second singular value $\sigma_2(W_{n,N})$ grows as $O(N^{2n-3})$ and the corresponding left and right singular vectors tend to $e_2 = [0 \quad 1 \cdots 0]$, etc. Let us now prove this.

We start by writing $W_{n,N}$ as

$$W_{n,N} = \begin{bmatrix} \displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2 & \left(\displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} A_{n-1}^{J} e_{\text{last}} \right)^T \\ \displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} A_{n-1}^{J} e_{\text{last}} & W_{n-1,N} \end{bmatrix}$$

Applying a congruence on $W_{n,N}$ with

$$Q = \begin{bmatrix} 1 & 0 \\ -\dfrac{\displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} A_{n-1}^J e_{\text{last}}}{\displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2} & I \end{bmatrix}$$

we get

$$QW_N Q^T = \begin{bmatrix} \displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2 & 0 \\ 0 & \hat{W}_{n-1,N} \end{bmatrix}$$

where

$$\hat{W}_{n-1,N} = W_{n-1,N} - \frac{\left(\displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} A_{n-1}^J e_{\text{last}}\right)\left(\displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix} A_{n-1}^J e_{\text{last}}\right)^T}{\displaystyle\sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2}$$

Using routine algebraic manipulations, it can be shown that $\hat{W}_{n-1,N}$ is approximately

$$\begin{bmatrix} \frac{1}{2n-2} & & & \\ & \frac{2}{2n-3} & & \\ & & \ddots & \\ & & & \frac{n-1}{n} \end{bmatrix} W_{n-1,N} \begin{bmatrix} \frac{1}{2n-2} & & & \\ & \frac{2}{2n-3} & & \\ & & \ddots & \\ & & & \frac{n-1}{n} \end{bmatrix}$$

for large N.

We now observe that the congruence matrix $Q \rightarrow I$ as $N \rightarrow \infty$, implying that the maximum singular value $\sigma_1(W_{n,N}) \approx \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-1 \end{pmatrix}^2$ for large N, and that the corresponding singular vector converges to the first Euclidean basis vector e_1. Applying the block diagonalization technique recursively to $W_{n-1,N}$, $\ldots W_{1,N}$, we conclude that the ith singular value of $W_{n,N}$

$$\sigma_i(W_{n,N}) \approx \sum_{k=0}^{N-1} \begin{pmatrix} k \\ n-i \end{pmatrix}^2 \Big/ \begin{pmatrix} 2n-i \\ i-1 \end{pmatrix}^2$$

for large N, and the singular vectors tend to the the standard Euclidean basis of $\Re^n$, i.e.,

$$\{[1 \;\; 0 \cdots 0], [0 \;\; 1 \cdots 0], \ldots, [0 \;\; 0 \cdots 1]\}$$

Using (4.3), we may finally write

$$\sigma_i(W_{n,N}) \approx \left(\frac{(i-1)!\,(2n-2i+1)!}{(n-i)!\,(2n-i)!} \right)^2 \frac{1}{(2n-2i+1)} N^{2n-2i+1}$$

for large N, which concludes the proof. □

Figure 4.1 illustrates Theorem 9 for $n = 4$.

Corollary 1 *Consider the system*

$$x(k+1) = A_n^J x(k) + Bu(k) \tag{4.5}$$

where $B \in \Re^{n \times p}$ has a nonzero last row (so that the system is controllable). Theorem 9 holds for system (4.5) as well.

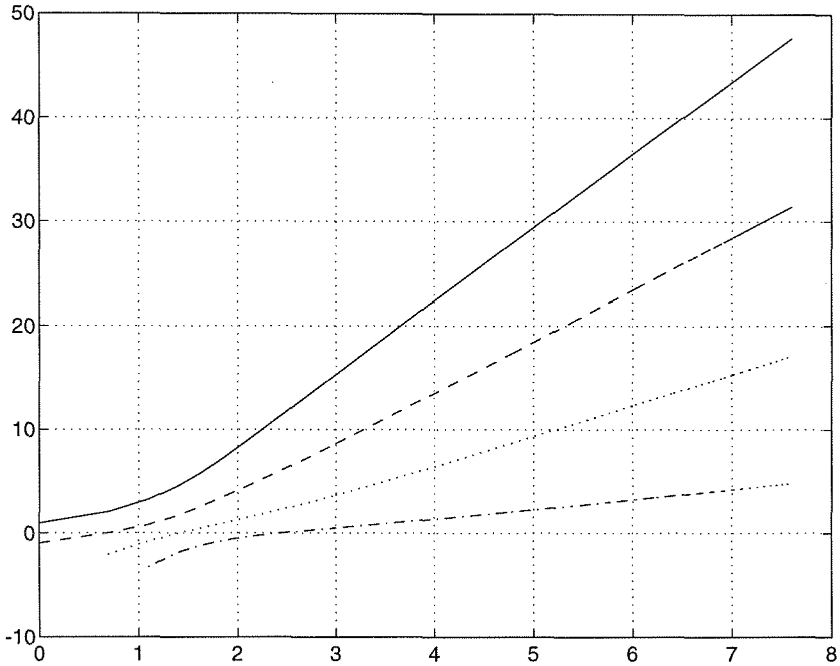

Figure 4.1: Logarithms of singular values of $W_{4,N}$ versus N

Proof. Let $b_1^T,\ b_2^T, \ldots, b_n^T$ be the rows of B, so that $B^T = [b_1\ b_2\ \cdots\ b_n]$. Then,

$$\begin{aligned}
\tilde{W}_{n,N} &= \sum_{k=0}^{N-1} (A_n^J)^k \begin{bmatrix} b_1^T b_1 & b_1^T b_2 & \cdots & b_1^T b_n \\ b_2^T b_1 & b_2^T b_2 & \cdots & b_2^T b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_n^T b_1 & b_n^T b_2 & \cdots & b_n^T b_n \end{bmatrix} ((A_n^J)^T)^k \\
&= \sum_{1 \leq i,j \leq n} b_i^T b_j \sum_{k=0}^{N-1} (A_n^J)^k e_i e_j ((A_n^J)^T)^k
\end{aligned}$$

Since

$$(A_n^J)^k e_j = \left[\begin{pmatrix} k \\ j-1 \end{pmatrix} \begin{pmatrix} k \\ j-2 \end{pmatrix} \cdots \begin{pmatrix} k \\ j-n \end{pmatrix} \right]^T, \quad j = 1, 2, \ldots, n$$

we have

$$\sum_{1\leq i,j\leq n} b_i^T b_j \sum_{k=0}^{N-1} (A_n^J)^k e_i e_j ((A_n^J)^T)^k \approx b_n^T b_n \sum_{k=0}^{N-1} (A_n^J)^k e_n e_n ((A_n^J)^T)^k$$

for large N, and the claim made in the corollary follows (recall that $b_n \neq 0$). □

Next, consider the system

$$x(k+1) = TA_n^J T^{-1} x(k) + TBu(k) \tag{4.6}$$

where $B \in \Re^{n\times p}$ has a nonzero last row. Let the N-step reachability Gramian of the pair $(TA_n^J T^{-1}, TB)$ be denoted by $\overline{W}_{n,N}$. We then have the following theorem.

Theorem 10 *Let $T = QR$ be the QR-factorization of T, i.e., Q is orthogonal and R is upper triangular with positive diagonal entries. Then, the singular values of $\overline{W}_{n,N}$ grow as*

$$\{O(N^{2n-1}), O(N^{2n-3}), \ldots, O(N)\}$$

Moreover, the matrix whose columns comprise the singular vectors of $\overline{W}_{n,N}$ converges to Q.

Proof. The N-step reachability Gramian $\overline{W}_{n,N}$ of the pair $(TA_n^J T^{-1}, TB)$ equals $T\tilde{W}_{n,N}T^T$, where $\tilde{W}_{n,N}$ is the N-step reachability Gramian of the pair (A_n^J, B). Then $\overline{W}_{n,N} = QR\tilde{W}_{n,N}R^TQ^T$. A direct calculation shows that

$$R\tilde{W}_{n,N}R^T \approx \begin{bmatrix} R_{11} & & \\ & \ddots & \\ & & R_{nn} \end{bmatrix} \left(\sum_{k=0}^{N-1} (A_n^J)^k e_{\text{last}} e_{\text{last}}^T ((A_n^J)^T)^k \right) \begin{bmatrix} R_{11} & & \\ & \ddots & \\ & & R_{nn} \end{bmatrix}$$

for large N, where R_{ii} is the ith diagonal element of R. (This is a direct consequence of the fact that R is upper-triangular.) This completes the proof. □

Corollary 2 *The above results extend immediately to the case when the eigenvalue of the Jordan block is not unity, but equals $re^{j\theta}$ for some $\theta \in [0, 2\pi]$ and some $r > 1$.*

(In this case, $W_{n,N}$ is defined to be $\sum_{k=0}^{N-1}(A_n^J)^k e_{\text{last}} e_{\text{last}}^T((A_n^J)^*)^k$.) Then the singular values of $W_{n,N}$ grow as

$$\{O(r^{2N}N^{2n-1}), O(r^{2N}N^{2n-3}), \ldots, O(r^{2N}N)\}$$

with N.

Proof. Let $A_n^{J(\lambda)}$ be a Jordan block of size n with eigenvalue $\lambda = re^{j\theta}$. It is easy to show that $A_n^{J(\lambda)}$ is similar to λA_n^J. This fact, combined with Theorems 9 and 10 immediately yields the desired conclusion. □

Corollary 3 *Let*

$$A = \begin{bmatrix} A^{J(1)} & & & \\ & A^{J(2)} & & \\ & & \ddots & \\ & & & A^{J(m)} \end{bmatrix}$$

where $A^{J(i)}$ is a Jordan block of size ν_i and eigenvalue $\lambda_i = e^{j\theta_i}$ for $i = 1, \ldots, m$ with (A, B) being controllable. Then the minimum eigenvalue of the N-step reachability Gramian of the pair (A, B) is $O(N)$.

Proof. The proof is very similar to the proof of Theorem 9. For simplicity of exposition, we will demonstrate the proof for the special case when the size of each Jordan block is two (i.e., $\nu_i = 2$ for all i), and when $B_i = e_{\text{last}}$. The proof for the general case should be readily apparent.

We first perform a similarity transformation so that

$$A = \begin{bmatrix} \lambda_1 A_{\nu_1}^J & & & \\ & \lambda_2 A_{\nu_2}^J & & \\ & & \ddots & \\ & & & \lambda_m A_{\nu_m}^J \end{bmatrix}$$

and

$$B = \begin{bmatrix} \lambda_1 e_{\text{last}} \\ \lambda_2 e_{\text{last}} \\ \vdots \\ \lambda_m e_{\text{last}} \end{bmatrix}$$

(With some abuse of notation, we will use A and B to denote the state-space matrices in the new coordinate systems as well, in order to avoid proliferation of symbols.)

We follow this with another similarity transformation (in fact, a simple permutation similarity) so that

$$A = \begin{bmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ \Lambda\, \mathbf{1} \end{bmatrix}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ and $\mathbf{1}$ is a vector of length m with each component unity.

In this new coordinates, the N-step reachability Gramian W_N satisfies

$$W_N \approx \begin{bmatrix} \sum_{k=0}^{N-1} k^2 I & \sum_{k=0}^{N-1} kI \\ \sum_{k=0}^{N-1} kI & \sum_{k=0}^{N-1} I \end{bmatrix}$$

for large N. Using the block diagonalization technique in the proof of Theorem 9, it is straightforward to show that m singular values of W_N are $O(N^3)$ and the remaining m singular values of W_N are $O(N)$. $\square$

4.2.2 Controllability to the Origin with Bounded Inputs

Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k) \tag{4.7}$$

Since we may always perform a state coordinate transformation that puts A in its Jordan form, we may assume, without loss of generality that

$$A = \begin{bmatrix} A^{J(1)} & & & \\ & A^{J(2)} & & \\ & & \ddots & \\ & & & A^{J(m)} \end{bmatrix}$$

where $A^{J(i)}$ is a Jordan block of size ν_i and eigenvalue λ_i for $i = 1, \ldots, m$. For future reference, we partition B and x conformally as

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

We now consider the problem of controlling the state of system (4.7) to the origin with unit-energy inputs:

$$\text{Given } x(0)\text{, find } u \text{ with } \sum_{i=0}^{\infty} u(i)^T u(i) \leq 1 \text{ such that } \lim_{k\to\infty} x(k) = 0 \tag{4.8}$$

We will show that a necessary and sufficient condition for this is that $\{A, B\}$ is stabilizable and A has all its eigenvalues in the closed unit disk, that is $\rho(A) \leq 1$. Indeed, we will show that for every $x(0) \in \Re^n$, there exists N such that the following problem is feasible, if and only if $\{A, B\}$ is stabilizable and $\rho(A) \leq 1$:

$$\text{Given } x(0), \text{ find } u \text{ and } N \text{ with } \left(\sum_{k=0}^{N-1} |u(k)|^2\right)^{1/2} \leq 1 \text{ such that } x(N) = 0 \tag{4.9}$$

First let us assume that $\rho(A) \leq 1$. Indeed, we may as well assume that all the eigenvalues of A are on the unit circle: Any eigenvalue in the open unit disk is a *stable* eigenvalue, and the projection of the initial condition $x(0)$ on the eigenspace of this eigenvalue decays to zero exponentially, with zero input, and therefore we may "ignore" these eigenvalues. (If there is no eigenvalue on the unit circle, then the problem is trivially solved with zero input!)

The condition $x(N) = 0$ yields

$$0 = \left[B \; AB \; \cdots \; A^{N-1}B\right] \left[u(0)^T \; u(1)^T \; \cdots \; u(N-1)^T\right]^T + A^N x(0)$$

Then, we need

$$x(0) = -\left[A^{-1}B \; A^{-2}B \; \cdots \; A^{-N}B\right] \left[u(N-1)^T \; u(N-2)^T \cdots \; u(0)^T\right]^T$$

In other words, $x(0)$ must be reachable for the system

$$\tilde{x}(k+1) = A^{-1}\tilde{x}(k) - A^{-1}Bu(k)$$

with unit-energy u, over N time steps. Since every eigenvalue of A^{-1} is of the form $e^{j\theta}$ for some $\theta \in [0, 2\pi]$, it follows from Corollary 3 that this is so. Thus sufficiency of the condition $\rho(A) \leq 1$ is proved.

Conversely, let $\rho(A) > 1$. Without loss of generality, say $|\lambda_1| > 1$. Then it is quite easy to show that for every initial condition of the form $x(0) = [z_1^T \;\; 0]^T$ with $z_1^T W_c^{-1} z_1 > 1$, problem (4.9) is infeasible, where W_c is given as the unique solution to the Lyapunov equation

$$W_c - A^{J(1)} W_c (A^{J(1)})^T + B_1 B_1^T = 0$$

Thus, we have the following theorem.

Theorem 11 *For every $z \in \Re^n$, there exists N such that the system (4.7) is controllable from z to 0 in N time steps with unit-energy inputs if and only if $\rho(A) \leq 1$ and $\{A, B\}$ is controllable.*

Remark 9 *Since the set of reachable states grows linearly with the energy bound on the input, we note that the above claims hold for any* arbitrarily small *bound on the energy, not necessarily unity.*

Often, the following variation on problem (4.9) is of interest:

$$\text{Given } x(0), \text{ find } u \text{ and } N \text{ with } |u(k)|_\infty \leq 1,\ k = 0, \ldots, N-1, \text{ such that } x(N) = 0 \tag{4.10}$$

This problem concerns the controllability to the origin from $x(0)$ with *unit-peak* inputs, in contrast to the unit-energy inputs considered earlier.

It may be shown that problem (4.10) is feasible if and only if all the eigenvalues of A are in the closed unit disk. It follows immediately that the latter condition is sufficient for problem (4.10) to be feasible: the set of unit-peak inputs contains the set of unit-energy inputs.

The proof of necessity can be outlined as follows. Suppose that one of the eigenvalues of A is outside the unit circle. At the sampling time k, the value of the state has two contributions, one from the initial condition ($x(0)$) and the other from the controls (u) up to the sampling time $k-1$. For sufficiently large k, the contribution from the initial condition behaves as $\beta e^{\lambda k}$ where $\lambda > 0$ and β is a constant that depends on the initial condition and can be made *arbitrarily large* for some initial condition. The contribution from the control input at the sampling time $i < k$ behaves as $\gamma e^{\lambda(k-i)}$. Since the control input is bounded, γ is bounded. Simple calculations show that the total contribution from the controls up to the sampling time $k-1$ is bounded by $\bar{\gamma} e^{\lambda k}$ where $\bar{\gamma}$ is constant. Thus if we chose an initial condition such that $|\beta| > \bar{\gamma}$, then the output will grow unbounded regardless of control actions. Therefore, there are initial conditions that cannot be controlled to the origin, even with unit-peak inputs, if the controlled system has eigenvalues outside the unit disk. In other words, Theorem 11 may be extended to the case of unit-peak inputs:

Theorem 12 *For every $x(0) \in \Re^n$, there exists N such that the system (4.7) is controllable from $x(0)$ to 0 over N time steps with unit-peak inputs if and only if $\rho(A) \leq 1$ and (A, B) is controllable.*

Remark 10 *As before, the claim in Theorem 12 holds for any* arbitrarily small *bound on the peak, not necessarily unity.*

Remark 11 *Controllability of $\{A, B\}$ can be replaced by stabilizability of $\{A, B\}$ if we replace $x(N) = 0$ by $\lim_{k\to\infty} x(k) = 0$.*

4.3 Semi-Global Stabilization

In this section, we will prove that the IHMPCMC algorithm is stabilizing for any initial condition if the input horizon (H_c) is sufficiently long. Notice that H_c depends on the initial condition. This kind of stabilization is usually referred to in the literature as semi-global stabilization: Given any initial condition (or a set of initial conditions), there exists an H_c such that the controller stabilizes the initial condition (or the set of initial conditions) to the origin. On the other hand, there *does not* exist a constant H_c that will stabilize all initial conditions to the origin.

Define the objective function as follows:

$$\Phi_k = \sum_{i=1}^{\infty} |x(k+i|k)|^2_{\Gamma_x} + \sum_{i=0}^{H_c} \left[|u(k+i|k)|^2_{\Gamma_u} + |\Delta u(k+i|k)|^2_{\Gamma_{\Delta u}} \right] \tag{4.11}$$

where $\Gamma_x > 0, \Gamma_u > 0, \Gamma_{\Delta u} \geq 0$, and H_c is finite. Γ_x, Γ_u and $\Gamma_{\Delta u}$ are symmetric. $(\cdot)(k+i|k)$ denotes the variable $(\cdot)$ at sampling time $k+i$ predicted at sampling time k. The control actions are generated by *Controller IHMPCMC* which is defined as follows.

Definition 7 Controller IHMPCMC: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$*

which is the minimizer of the optimization problem

$$J_k = \min_{\epsilon(k),u(k|k),\cdots,u(k+H_c|k)} \quad \Phi_k + |\epsilon(k)|^2_{\Gamma_\epsilon}$$

$$\text{subject to} \quad \begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, 1, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, H_c + 1, \cdots, \infty \\ x(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0, 1, \cdots, \infty \end{cases} \tag{4.12}$$

where $\Gamma_\epsilon > 0$ *is diagonal, and*

$$\begin{aligned} \mathcal{U} &\triangleq \{u : 0 > u^{min} \leq u \leq u^{max} > 0\} \\ \mathcal{X}_\epsilon &\triangleq \left\{ x : [F_x \quad F_u] \begin{bmatrix} x \\ u \end{bmatrix} \leq f + \epsilon, \epsilon \geq 0, u \in \mathcal{U} \right\} \end{aligned}$$

We assume throughout this chapter that $\mathcal{U}$ contains $u = 0$ as an *interior* point and $\mathcal{X}$ contains $x = 0$ as an *interior* point. An important question associated with *Controller IHMPCMC* is that of stability: *Given* $x(0)$*, does* Controller IHMPCMC *always lead to a control* u *that steers the state to zero?*

We may break the answer to this question into two parts: First, we require $J_k < \infty$ for each k. If this condition is satisfied, we may then ask if the overall strategy—that of implementing as input only the first element of the minimizer at each step—is stable.

Obviously, $J_k < \infty$ for all $x(k) \in \Re^n$ if and only if for every $x(k)$, the projection of $x(k + H_c|k)$ on the eigenspace of A corresponding to the unstable (that is, with magnitude that is not less than one) eigenvalues is zero. The results of Section 4.2.2 immediately give us the following: for every $x(k)$, there exists a value of H_c such that $J_k < \infty$ if and only if (A, B) is stabilizable and all the eigenvalues of A are in the closed unit disk.

Next, let us consider the stability of the moving horizon strategy. First, if $J_k < \infty$

for some k, then $J_{k+1} < \infty$. Indeed $J_\cdot$ serves as a Lyapunov function that proves the stability of the horizon strategy. This can be seen as follows. Assuming $J_k < \infty$, let $\{u(k|k), u(k+1|k), \ldots, v(k+1|H_c-1|k)\}$ be the minimizer of problem (4.12). Then we have that for problem (4.12) at time $k+1$, the input $\{u(k+1|k), u(k+2|k), \ldots, u(k+H_c-1|k), 0\}$ leads to a finite objective that equals

$$J_k - \left(x(k)^T\Gamma_x x(k) + u(k)^T\Gamma_u u(k) + \Delta u(k)^T\Gamma_{\Delta u}\Delta u(k)\right)$$

Thus, if $J_k < \infty$, then $J_{k+1} < \infty$. Also,

$$J_k + x(k)^T\Gamma_x x(k) + u(k)^T\Gamma_u u(k) + \Delta u(k)^T\Gamma_{\Delta u}\Delta u(k) \leq J_k$$

which yields

$$J_k + \sum_{i=0}^{k-1}\left[x(i)^T\Gamma_x x(i) + u(i)^T\Gamma_u u(i) + \Delta u(i)^T\Gamma_{\Delta u}\Delta u(i)\right] \leq J_0 < \infty$$

for all $k > 0$, which, in turn, implies that $x(k) \to 0$ as $k \to \infty$. The above discussion is summarized in the following theorem.

Theorem 13 *The closed loop system with* Controller IHMPCMC *is semi-globally asymptotically stable for a sufficiently large finite H_c if and only if (A, B) is stabilizable and $\rho(A) \leq 1$.*

Thus, given $x(0)$, we conclude that *Controller IHMPCMC* is stabilizing for some input horizon H_c if and only if (A, B) is stabilizable and $\rho(A) \leq 1$.

4.4 Global Stabilization

In the previous section, we showed that the IHMPCMC algorithm *semi-globally* stabilizes a stabilizable system with poles on the unit disk. However, H_c depends on the initial condition; thus, it is generally difficult to determine *a priori* and can be arbitrarily large which implies demanding computations. Furthermore, in practice an unmeasured disturbance could still cause the optimization problem to become infeasible and an even larger H_c may have to be chosen. Therefore, the strategy is *not* easily implementable. In this section, we propose an implementable IHMPCMC algorithm and show that with this scheme a discrete-time linear system with n poles on the unit disk (with any multiplicity) can be globally stabilized if $H_c \geq n$. For the specific case of a chain of n integrators, this condition is also necessary. Furthermore, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input. For notational simplicity, all the results in this section are proved for single-input single-output (SISO) systems. We discuss the extension of the results to multi-input multi-output (MIMO) systems.

4.4.1 Preliminary

Systems

The system which we will consider here is linear time-invariant discrete-time with poles on the unit circle and can be represented generally as follows.

$$\left[(1-q^{-1})^{n_0}(1+q^{-1})^{n_1}\prod_{i=2}^{n_a}(1+2a_iq^{-1}+q^{-2})^{n_i}\right]y(k) = \left[\sum_{i=1}^{n_b} b_i q^{-i}\right]u(k) \tag{4.13}$$

where $n_i, i = 0, 1, \cdots, n_a$, and n_b are integers, q^{-1} is the backward-shift operator, and $|a_i| < 1, i \geq 2$. The term $(1-q^{-1})^{n_0}$ represents n_0 integrators, $(1+q^{-1})^{n_1}$ n_1 poles at -1 and $(1+2a_iq^{-1}+q^{-2})^{n_i}$ n_i pairs of complex conjugate poles at $-a_i \pm \sqrt{a_i^2-1}$. Assume that the left-hand and right-hand polynomials of (4.13) do not have any common roots. Define

$$\begin{aligned}
n &= n_0 + n_1 + 2\sum_{i=2}^{n_a} n_i \\
n_{\max} &= \max_{0\leq i\leq n_a} n_i \\
n_{\text{modes}} &= \min(n_0, 1) + \min(n_1, 1) + 2\sum_{j=2}^{n_a} \min(n_j, 1)
\end{aligned}$$

Here n is the total number of poles on the unit disk, $n_{\max}$ is the largest multiplicity, n_{modes} is the total number of poles on the unit disk not counting multiplicity. The unforced response, *i.e.* $u(k) = 0,\ \forall\ k \geq 0$, is

$$y(k) = \sum_{i=1}^{n_{\max}} P_i(k)Q_i \quad \forall\ k \geq n_b \tag{4.14}$$

where $P_i(k) = [1\ \cos(\pi k)\ \sin(\omega_2 k)\ \cos(\omega_2 k)\ \cdots\ \sin(\omega_{n_a} k)\ \cos(\omega_{n_a} k)]k^{i-1}$,[1] $\omega_j = \arccos(-a_j) \in (0, \pi), j \geq 2$, and Q_i is a constant column vector that depends on the initial condition $y_0 = [y(-n+n_b)\ \cdots\ y(n_b - 1)]$. Let $Q = \begin{bmatrix} Q_1 \\ \vdots \\ Q_{n_{\max}} \end{bmatrix}$ and $P(k) = [P_1(k)\ \cdots\ P_{n_{\max}}(k)]$, then we have $y(k) = P(k)Q$.

Example 2 Consider the system

$$(1 + q^{-1})(1 - q^{-1} + q^{-2})^2 y(k) = u(k-1)$$

with the initial condition

$$y_0 = [y(-4)\ y(-3)\ y(-2)\ y(-1)\ y(0)].$$

Then $n = 5$, $n_{\text{modes}} = 3$, $n_{\max} = 2, \omega_2 = \arccos(0.5) = \frac{\pi}{3}, P_1(k) = [\cos(\pi k)\ \sin(\omega_2 k)\ \cos(\omega_2 k)]$, and $P_2(k) = [\sin(\omega_2 k)k\ \cos(\omega_2 k)k]$. Q can be calculated using the relationship

$$y_0 = [P(-4)^T\ \cdots\ P(0)^T]^T Q \equiv D_0 Q.$$

Notice that D_k is not singular for all k. Otherwise, there would be some coefficients that do not depend on the initial condition.

[1]Since n_i is not necessarily equal to $n_{\max}$ for all $0 \leq i \leq n_a$, P_i may not contain every term shown here. For example, if $n_0 = 0$, then $P_i(k)$ does not contain the constant term 1 for all $i \geq 1$.

Objective function

Consider the following objective function:

$$\Phi(k,0) = \sum_{i=1}^{\infty} |r - y(k+i|k)|^2 + \Gamma_u \sum_{i=0}^{H_c-1} |\Delta u(k+i|k)|^2 \qquad 0 \leq \Gamma_u < \infty \qquad (4.15)$$

Since the system (4.13) contains poles on the unit disk and the input is constrained, H_c must be sufficiently large, as shown in the previous section, to bring the steady-state to the setpoint. However, for computational reasons we would like to keep H_c small and for small H_c the value of the objective function (4.15) may be unbounded. We want to modify the objective function such that it is bounded for all values of H_c. For systems with poles on the unit disk, the state may grow at most as $k^{n_{max}-1}$. Multiplying the objective function (4.15) by the term $\frac{1}{p^\beta}$ and choosing β appropriately will make the objective function bounded for all values of H_c. This motivates the following *modified* objective function.

$$\Phi(k,\alpha) = \lim_{H_p \to \infty} \frac{1}{H_p^{\beta(\alpha)}} \left[\sum_{i=1}^{H_p} |r - y(k+i|k)|^2 + \Gamma_{\Delta u} \sum_{i=0}^{H_c-1} |\Delta u(k+i|k)|^2 \right] \qquad (4.16)$$

where $\beta(\alpha) = \max(2\alpha - 1, 0)$ and α is the smallest nonnegative integer such that the optimal value of the objective function is finite.

Remark 12 *The poles inside the unit disk do not affect $\Phi(k,\alpha)$, $\alpha \geq 1$. This is because $\sum_{i=1}^{\infty} |y_s(k+i|k)|^2$, where y_s denotes the output contribution from the poles inside the unit disk, is finite.*

Remark 13 *The modified objective function (4.16) can be extended directly to handle MIMO systems as follows.*

$$\Phi(k,\alpha) = \lim_{H_p \to \infty} \frac{1}{H_p^{\beta(\alpha)}} \left[\sum_{i=1}^{H_p} |r - y(k+i|k)|_{\Gamma_y}^2 + \sum_{i=0}^{H_c-1} |\Delta u(k+i|k)|_{\Gamma_{\Delta u}}^2 \right] \qquad (4.17)$$

where $\beta(\alpha) = \max(2\alpha - 1, 0)$ and α is the smallest nonnegative integer such that the optimal value of the objective function is finite.

Control design

At each sampling time, H_c control moves are calculated such that $\Phi(k, \alpha)$ is minimized where α is the smallest integer such that the optimal value of $\Phi(k, \alpha)$ is finite. The value of α can be determined as follows: since the optimal output grows at most as $k^{n_{max}-1}$, $J(k, n_{max} + i) = 0, \ \forall \ i \geq 1$. Starting with the initial guess n_{max} for α, we reduce the value of α by one until $J(k, \alpha) > 0$. The optimal control moves are generated by *Implementable Controller IHMPCMC* which is defined below.

Definition 8 Implementable Controller IHMPCMC: *At each sampling k, the control moves $u(k)$ is determined as follows.*

Step 1 *Set $\alpha = n_{max}$.*

Step 2 *Solve the following optimization problem.*

$$J(k, \alpha) = \min_{U_k} \Phi(k, \alpha)$$
$$\text{subject to} \begin{cases} u^{min} \leq u(k+i|k) \leq u^{max}, & i = 0, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max}, & i = 1, \cdots, H_c - 1 \\ \Delta u(k+i|k) = 0, & i = H_c, \cdots, \infty \\ \Phi(k, \alpha + i) = 0, & i = 1, \cdots, n_{max} - \alpha \end{cases} \tag{4.18}$$

where $U_k = [u(k|k) \ \cdots \ u(k + H_c - 1|k)]^T$.

Step 3 *If $J(k, \alpha) = 0$ and $\alpha \geq 1$, then set $\alpha = \alpha - 1$ and go to Step 2. Otherwise, go to Step 4.*

Step 4 *Set the control moves $u(k)$ equal to the first element $u(k|k)$ of the sequence $\{u(k|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem (4.18).*

Notice that $\Phi(k, \alpha + i) = 0, i = 1, \cdots, n_{max} - \alpha$, is necessary to ensure that α is the smallest integer for which the optimal value of the objective function is finite.

Remark 14 *Here we did not include the soft output constraints for simplicity* only. *Inclusion of such constraints does not affect any results to be presented later.*

Remark 15 *In the absence of disturbances, the value of α does not increase with time. The value of α at time k can be determined by starting with the value at time $k-1$ as the initial guess. However, in practice, because of disturbances and/or model/plant mismatch, the value of α at each sampling time must be determined by starting with the initial guess n_{max}.*

4.4.2 Main Results

The infinite-horizon minimization problem is converted into a finite-dimensional optimization via the following lemma.

Lemma 2 *Suppose $r = 0$. Assume that at sampling time k, the coefficients (Q) are calculated by treating $k + H_c + n_b - 1$ as the initial time. Clearly, Q depends on U_k. Then $J(k,\alpha)$ is finite if and only if $Q_i = 0,\ i \geq \alpha+1$. Moreover, if $\alpha \neq 0$ and $Q_i = 0,\ i \geq \alpha+1$, then $J(k,\alpha) = \min_{U_k} Q_\alpha^T W_\alpha Q_\alpha$ where $W_\alpha = \frac{1}{2\alpha-1}\mathrm{diag}\{1, 1, \frac{1}{2}, \cdots, \frac{1}{2}\}$.*

Proof. If $Q_{\alpha+1} \neq 0$, then the output grows as $O(k^\alpha)$. $\lim_{H_p\to\infty} \frac{1}{H_p^\beta} \sum_{k=1}^{H_p} |\mathcal{O}(k^\alpha)|^2$ clearly approaches infinity for all $\alpha \geq 0$. If $Q_i = 0, \forall\, i \geq 1$, then $J(k,0)$ is clearly finite. The sufficiency for $\alpha \geq 1$ follows by establishing the second part of the lemma which we do now.

Since the output horizon is infinite, the term $P_\alpha(k)Q_\alpha$ in the output which grows as $O(k^{\alpha-1})$ dominates. The second term in the objective function also vanishes. WLOG, assume that k is chosen such that $u(k) = 0, k \geq 0$.[2] Then by Equation (4.14), we have

$$
\begin{aligned}
\Phi(k,\alpha) &= \lim_{H_p\to\infty} \frac{1}{H_p^{2\alpha-1}} \sum_{k=1}^{H_p} [P_\alpha(k)Q_\alpha]^2 \\
&= \lim_{H_p\to\infty} \frac{1}{H_p^{2\alpha-1}} \sum_{k=1}^{H_p} Q_\alpha^T P_\alpha^T(k) P_\alpha(k) Q_\alpha
\end{aligned}
$$

[2]In the presence of the disturbance w entering at the plant input, $u(k) + w = 0$.

$$= \quad Q_\alpha^T W Q_\alpha$$

where

$$W = \lim_{H_p\to\infty} \frac{1}{H_p^{2\alpha-1}} \sum_{k=1}^{H_p} P_\alpha^T(k) P_\alpha(k)$$

$$= \lim_{H_p\to\infty} \frac{1}{H_p^{2\alpha-1}} \sum_{k=1}^{H_p} \begin{bmatrix} 1 \\ \cos(\pi k) \\ \sin(\omega_2 k) \\ \cos(\omega_2 k) \\ \vdots \\ \cos(\omega_{n_a} k) \end{bmatrix} [1 \ \cos(\pi k) \ \sin(\omega_2 k) \ \cos(\omega_2 k) \ \cdots \ \cos(\omega_{n_a} k)]\, k^{2\alpha-2}$$

$$= \lim_{H_p\to\infty} \frac{1}{H_p^{2\alpha-1}} \sum_{k=1}^{H_p} \begin{bmatrix} 1 & \cos(\pi k) & \cdots & \cos(\omega_{n_a} k) \\ \cos(\pi k) & \cos(\pi k)^2 & \cdots & \cos(\pi k)\cos(\omega_{n_a} k) \\ & & \vdots & \\ \cos(\omega_{n_a} k) & \cos(\pi k)\cos(\omega_{n_a} k) & \cdots & \cos(\omega_{n_a} k)^2 \end{bmatrix} k^{2\alpha-2}$$

$$= \frac{1}{2\alpha-1} \mathrm{diag}\{1, 1, \frac{1}{2}, \cdots, \frac{1}{2}\}$$

The last equality follows from the following integrals.

$$\int_1^{H_p} k^{2\alpha-2} \sin(\omega_1 k)\cos(\omega_2 k) dk \quad \sim \quad O(H_p^{2\alpha-2}) \quad \text{for large } H_p$$

$$\int_1^{H_p} k^{2\alpha-2} \sin(\omega_1 k)\sin(\omega_2 k) dk \quad \sim \quad \begin{cases} O(H_p^{2\alpha-1}) & \text{if } \omega_1 = \omega_2 \\ O(H_p^{2\alpha-2}) & \text{if } \omega_1 \neq \omega_2 \end{cases} \quad \text{for large } H_p$$

$$\int_1^{H_p} k^{2\alpha-2} \cos(\omega_1 k)\cos(\omega_2 k) dk \quad \sim \quad \begin{cases} O(H_p^{2\alpha-1}) & \text{if } \omega_1 = \omega_2 \\ \mathcal{O}(H_p^{2\alpha-2}) & \text{if } \omega_1 \neq \omega_2 \end{cases} \quad \text{for large } H_p$$

$\square$

Remark 16 *If $r \neq 0$ is such that the steady-state input is strictly within the constraints, the lemma still holds. By change of variables, the desired output becomes the origin and Q must be determined using the values for the new variables.*

Remark 17 *W_α may not contain every term shown. For example, for the system considered in Example 2, $W_1 = \mathrm{diag}\{1, \frac{1}{2}, \frac{1}{2}\}$ and $W_2 = \frac{1}{3}\mathrm{diag}\{\frac{1}{2}, \frac{1}{2}\}$. If we used the L_2-norm $(\int_0^{H_p} |y(k+t|k)|^2 dt)$ instead of the l_2-norm $\left(\sum_{i=1}^{H_p} |y(k+i|k)|^2\right)$, then $W_\alpha = \frac{1}{2\alpha-1}\mathrm{diag}\{1, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\}$.*

Remark 18 *One difficulty may arise in extending this lemma to MIMO systems. The order of growth for each output may be different. For example, one output may grow as $O(k^2)$ while another one may grow as $O(k^4)$. Therefore, different values of α may have to be used for each output.*

Remark 19 *For $\alpha \geq 1$, the solution to the optimization problem (4.18) may* not *be unique. If this is the case, we assume that the unique solution is such that*

$$\left\| \begin{bmatrix} u(k) \\ U_{k+1} \end{bmatrix} - \begin{bmatrix} U_k \\ u(k+H_c|k) \end{bmatrix} \right\|_2^2$$

is minimized over all feasible control moves for which the objective function has the optimal value.

The following theorem establishes a necessary condition and a sufficient condition on H_c such that the closed-loop system is globally asymptotically stable with *Implementable Controller IHMPCMC*. The proof of this theorem is lengthy and can be found in Section 4.4.7.

Theorem 14 *Suppose that a disturbance w enters at the plant input and that the disturbance has the following properties:*

1. $w(k) \rightarrow \bar{w}$ *as* $k \rightarrow \infty$ *and* $-\bar{w}$ *is strictly within the input limits,* i.e. $u^{min} - u_r^{ss} < -\bar{w} < u^{\max} - u_r^{ss}$ *where* u_r^{ss} *is the steady-state input resulting from the setpoint change* r.

2. For any $\epsilon > 0$, *there exists a finite* K *such that* $|w(k+1) - \bar{w}| < \epsilon \; \forall \; k \geq K$.

The future disturbance is estimated by assuming that it is a step. Then the closed-loop system with Implementable Controller IHMPCMC *is globally asymptotically stable,* i.e. $y(k) \rightarrow r$ *as* $k \rightarrow \infty$, *if* $H_c \geq n+1$ *and only if* $H_c \geq n - n_{\text{modes}} + 2$ *where* n *is the total number of poles (with any multiplicity) on the unit disk.*

Proof. See Section 4.4.7. □

For pure integrator systems, $n_{\text{modes}} = 1$ and the following corollary is immediate.

Corollary 4 *Under the conditions of Theorem 14, the closed-loop system with* Implementable Controller IHMPCMC *is globally asymptotically stable if and only if* $H_c \geq n+1$ *for pure integrator systems.*

In the absence of the disturbance, we have the following corollary.

Corollary 5 *In the absence of the disturbance,* $J(k, \alpha) = 0 \; \forall \; \alpha \geq 1$ *for a sufficiently large finite* H_c.

This corollary implies that for a sufficiently large number of control moves, the original objective function (4.15) is finite. Thus this result parallels those in the previous section and those in the paper by Tsirukis and Morari [92].

4.5 Examples

We have shown that, with H_c properly chosen, the IHMPCMC algorithms can semi-globally or globally stabilize *any* constrained stabilizable system with poles on the unit disk. Example 3 compares the closed loop responses for *Controller IHMPCMC* with other design methods. Example 4 illustrates how to choose H_c for *Implementable Controller IHMPCMC* to reach the best compromise between performance and computational complexity.

Example 3 [92] Consider the following system [87]

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + x_3\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= -x_3 + u\end{aligned}$$

where u must satisfy the constraint $|u| \leq 1$. The system has four poles on the imaginary axis $(-j, -j, j, j)$. As shown by Teel [91], no linear controller can globally stabilize this system.

The system was discretized with a sampling time of 0.1. The initial condition is $x_0 = [1 \;\; 0.5 \;\; 0.5 \;\; 1]^T$. The weights are $\Gamma_x = I, \Gamma_u = 10$, and $\Gamma_{\Delta u} = 0$. The input horizon is $H_c = 50$. Figure 4.2 depicts the time-evolution of state x_1 for the controller designed by Sontag and Yang [87] and *Controller IHMPCMC*. The behavior of the other three states is similar. The corresponding control actions are shown in Figure 4.3. Although both controllers stabilize the system, the difference in performance is striking. In all fairness, we should point out that the controller designed by Sontag and Yang [87] was to ensure stability and that they made no attempt to achieve good performance.

Example 4 [89] Consider the following triple-integrator system.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= u\\ y &= x_1\end{aligned} \tag{4.19}$$

As shown by Teel [91], no linear controller can globally stabilize this system. We discretize the system with a sampling time of 0.1. The initial condition is $x(0) = [3 \;\; -1 \;\; 3]^T$ and the control input is constrained between the saturation limits ± 1. To

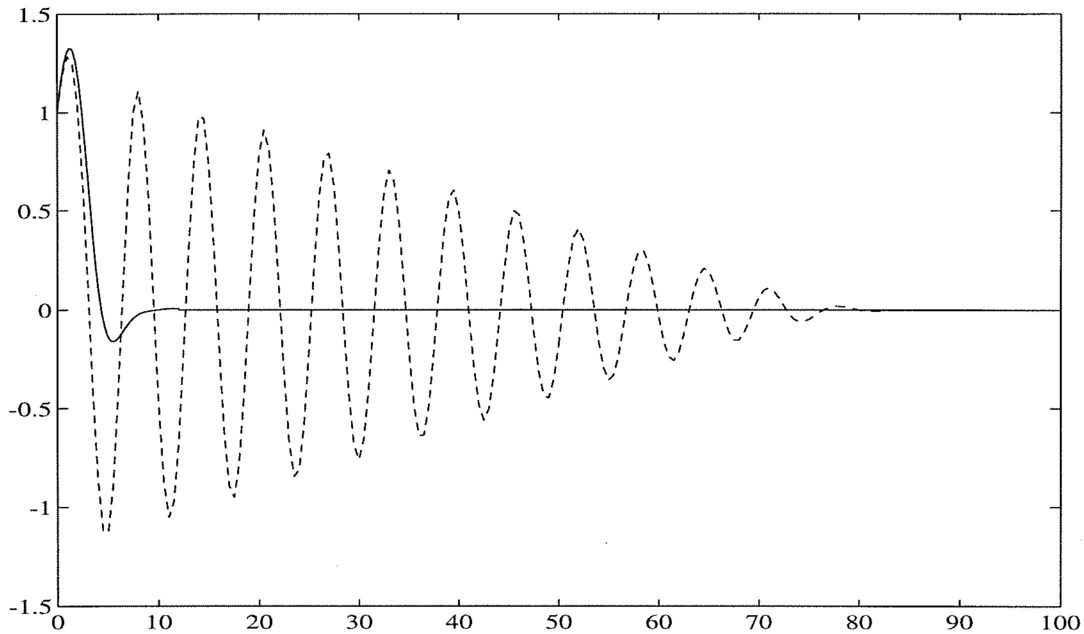

Figure 4.2: Time-evolution of x_1 for Example 3 (solid – MPC; dash – from Sontag and Yang 1991)

stabilize this initial condition with *Controller IHMPCMC*, we must choose $H_c > 150$. Theorem 14 states that with *Implementable Controller IHMPCMC* $H_c = 4$ is sufficient to globally stabilize this system. Figure 4.4 shows the responses for $H_c = 4, 10, 20, 40$, and 60 along with the response for the nonlinear controller designed by Sussmann et al. [90]. The input weight is $\Gamma_{\Delta u} = 0$. As we can see, the performance improves as the input horizon (H_c) increases. However, the amount of computation increases *dramatically*.[3] Thus a trade-off between performance and computation arises. Although Theorem 14 states that $H_c = 4$ is sufficient to globally stabilize this system, H_c should be chosen to reach the best compromise between performance and computation.

[3]It was observed that computational time grew exponentially in H_c.

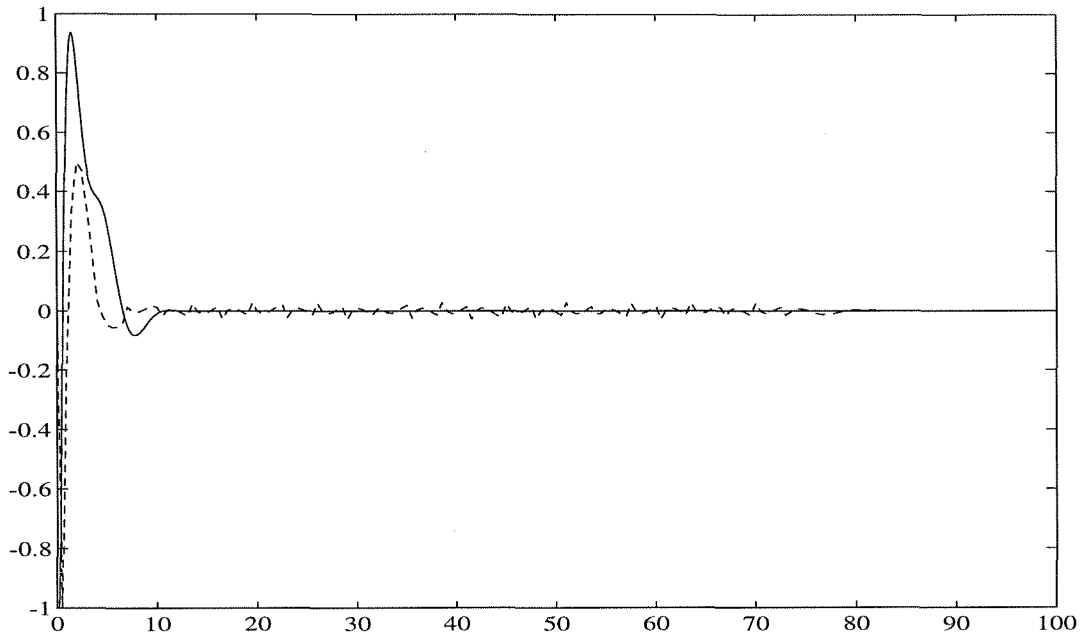

Figure 4.3: Time-evolution of control action for Example 3 (solid – MPC; dash – from Sontag and Yang 1991)

4.6 Conclusions

Based on the growth rate of the set of states reachable with unit-energy inputs, we showed that a discrete-time controllable linear system is globally controllable to the origin with energy-bounded inputs if and only if all its eigenvalues lie in the closed unit disk. These results imply that, with proper choice of the input horizon, the IHMPCMC algorithm is semi-globally stabilizing if and only if the controlled system is stabilizable and all its eigenvalues lie in the closed unit disk.

The disadvantage of the IHMPCMC algorithm is that the input horizon necessary for stabilization depends on the initial condition and can be arbitrarily large. As a result, we propose an implementable IHMPCMC algorithm. We show that with this algorithm a discrete-time linear system with n poles on the unit disk (with any multiplicity) can be globally stabilized if the input horizon is larger than n. For pure integrator systems, this condition is also necessary. Moreover, we show that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input.

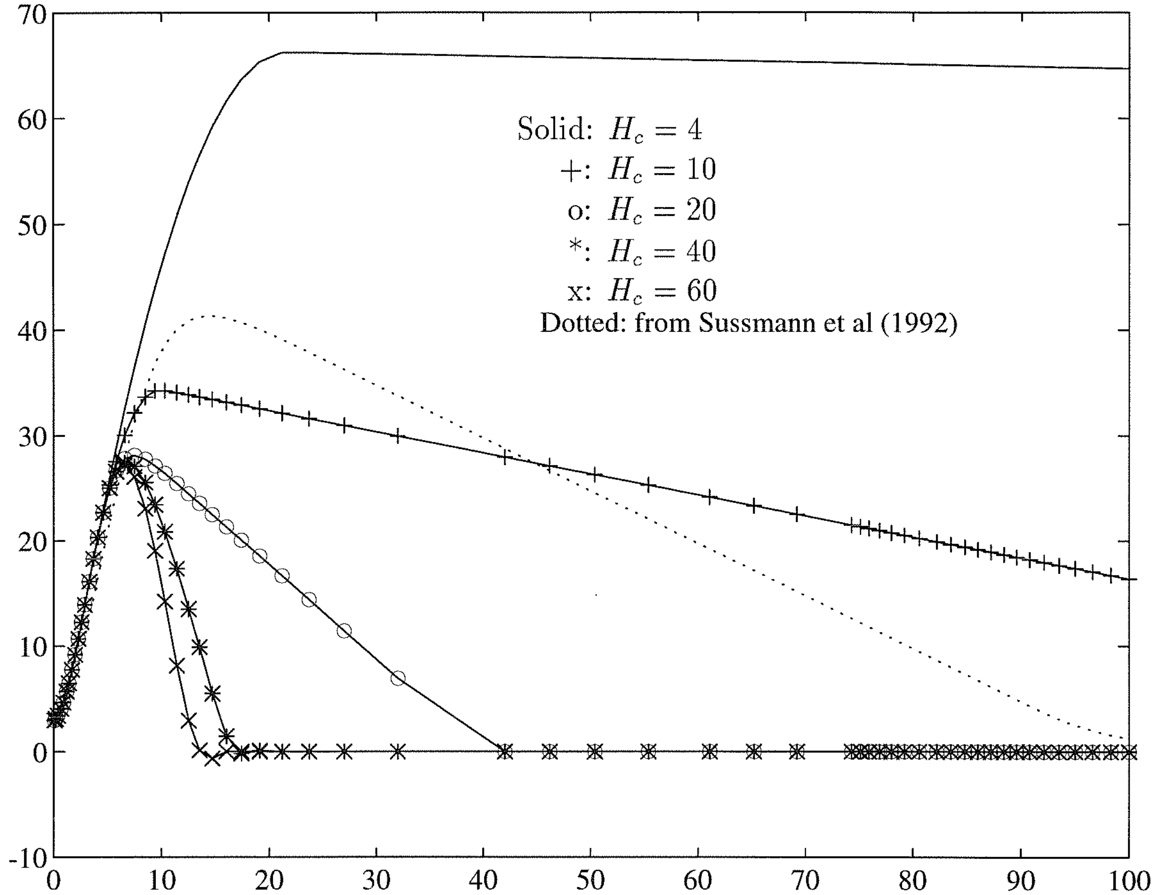

Figure 4.4: Output responses for various H_c values

4.7 Appendix–Proof of Theorem 14

Before we prove Theorem 14, let us first establish some preliminary results.

Claim 1 *Let $V \in \Re^{m\times m}$ be a unitary matrix. $z_2^{opt} = \arg\min_{z_2} z_2^T z_2$ subject to*

$$0 \geq x^{min} \leq V \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leq x^{\max} \geq 0 \tag{4.20}$$

where $z_2 \in \Re^{m_2}, m > m_2, x^{min} \in \Re^m$ *and* $x^{\max} \in \Re^m$. *There exists a positive constant* λ *such that*

$$z_1^T z_1 \geq \lambda (z_2^{opt})^T z_2^{opt} \quad \text{for all feasible } z_1{}^4$$

Proof. If $z_2^{opt} = 0$, the claim clearly holds. Assume that $z_2^{opt} \neq 0$. Then the optimal solution must occur on the boundary. The feasible region formed by the constraints (4.20) has $m2^{m-1}$ edges (or lines). Each edge is represented by

$$V_1 \begin{bmatrix} z_1 \\ \\ z_2 \end{bmatrix} = x$$

where V_1 consists of $m-1$ rows of V and x consists corresponding rows from either x^{min} or $x^{\max}$. After eliminating $m-2$ variables (only one variable in z_1 and one variable in z_2 remain), we obtain

$$\mu_i z_1(i) + \nu_j z_2(j) = c_{ij} \quad i = 1, \cdots, m - m_2 \quad \text{and} \quad j = 1, \cdots, m_2$$

If $\mu_i = 0$, then any change in $z_1(i)$ does not affect $z_2(j)$ and $z_2(j)^{opt} = 0$ since it is feasible. Let λ be the smallest value of $\min_{i,j,\mu_i \neq 0} \frac{|\mu_i|}{|\nu_j|}$ over all edges. We have $\lambda (z_2^{opt})^T z_2^{opt} \leq z_1^T z_1$ for all edges where the optimal solution lies. If the optimal solution does not occur on any edge, then some of the constraints are not satisfied as equalities and the value of $(z_2^{opt})^T z_2^{opt}$ must be smaller. Thus, we have

$$z_1^T z_1 \geq \lambda (z_2^{opt})^T z_2^{opt} \quad \text{for all feasible } z_1$$

where λ is a positive constant. □

Claim 2 *Let* X *be a closed convex set. Suppose the point* x_0 *lies outside* X. *Then there is a plane that strictly separates* X *from* x_0.[3]

Claim 3 *Let* $J = \min_{x \in X} (x_0 - x)^T W (x_0 - x)$ *where* $W > 0$, X *is a closed convex set and* $0 \in X$. *Suppose that* x^{opt} *is the optimal solution. Then* $J \leq x_0^T W x_0 - (x^{opt})^T W x^{opt}$.

---

[4] z_2^{opt} clearly depends on z_1.

Proof. WLOG, assume that W is the identity matrix, *i.e.* $J = (x_0 - x^{opt})^T(x_0 - x^{opt})$. If $x_0 \in X$, *i.e.* $x^{opt} = x_0$, then $J = 0$ and the claim clearly holds. Suppose x_0 lies outside X. By Claim 2, there is a plane that strictly separates X from x_0. Let P be the separating plane that is orthogonal to the line passing through the points x_0 and x^{opt} and contains the point x^{opt}. Since the origin belongs to the set X, there exists another plane P' which contains the origin and is parallel to P. Let the intersection of the plane P' and the line passing through the points x_0 and x^{opt} be y. Since x_0, x^{opt} and y form one line and x^{opt} is between x_0 and y, $(x_0 - x^{opt})^T(x^{opt} - y) \geq 0$. Since both the origin and y belong to P' and the line passing through the points x_0, x^{opt} and y is perpendicular to P', $(x_0 - y)^T(y - 0) = 0$ and $(x^{opt} - y)^T(y - 0) = 0$, *i.e.* $x_0^T y = y^T y$ and $(x^{opt})^T y = y^T y$. We have

$$\begin{aligned} (x_0 - y)^T(x_0 - y) + y^T y = x_0^T x_0 + 2y^T y - 2x_0^T y &= x_0^T x_0 \\ (x^{opt} - y)^T(x^{opt} - y) + y^T y &= (x^{opt})^T x^{opt} \end{aligned}$$

Thus,

$$\begin{aligned} x_0^T x_0 - (x^{opt})^T x^{opt} &= (x_0 - y)^T(x_0 - y) - (x^{opt} - y)^T(x^{opt} - y) \\ &= (x_0 - x^{opt} + x^{opt} - y)^T(x_0 - x^{opt} + x^{opt} - y) \\ &\quad -(x^{opt} - y)^T(x^{opt} - y) \\ &= (x_0 - x^{opt})^T(x_0 - x^{opt}) + 2(x_0 - x^{opt})^T(x^{opt} - y) \\ &\geq (x_0 - x^{opt})^T(x_0 - x^{opt}) \\ &= J \end{aligned}$$

$\Rightarrow J \leq x_0^T x_0 - (x^{opt})^T x^{opt}$. $\square$

The following claim is a generalization of the previous claim.

Claim 4 *Let* $J = \min_{x \in X}(a_0 + Ex)^T W(a_0 + Ex)$ *where* $X = \{x : x \in \Re^m, Gx = 0, 0 \leq x^{min} \leq x \leq x^{\max} \geq 0\}$, $W \in \Re^{n \times n} > 0$, *and* $m \geq n$. $\begin{bmatrix} E \\ G \end{bmatrix}$ *has full row rank. If the* *solution is not unique, the optimal solution (x^{opt}) is determined as* $\arg\min x^T x$ *over all feasible solutions for which J has the optimal value. Then there exists a positive constant γ such that $J \leq x_0^T W x_0 - \gamma (x^{opt})^T x^{opt}$.*

Proof. Let $\begin{bmatrix} E \\ G \end{bmatrix} = \begin{bmatrix} U_E \\ U_G \end{bmatrix} \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^T$ where $\begin{bmatrix} U_E \\ U_G \end{bmatrix}$ and V^T are unitary matrices and Σ contains all the singular values. Since $\begin{bmatrix} E \\ G \end{bmatrix}$ has full row rank, $\Sigma > 0$. Let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V^T x$. The optimization problem becomes

$$J = \min_{z_1} (a_0 + U_E \Sigma z_1)^T W (a_0 + U_E \Sigma z_1) \tag{4.21}$$

subject to

$$\begin{cases} U_G \Sigma z_1 = 0 \\ 0 \geq x^{min} \leq V \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leq x^{\max} \geq 0 \end{cases}$$

For any given z_1^{opt}, $z_2^{opt} = \arg\min_{z_2} z_2^T z_2$ subject to $0 \geq x^{min} \leq V \begin{bmatrix} z_1^{opt} \\ z_2 \end{bmatrix} \leq x^{\max} \geq 0$ and z_1^{opt} is such that the constraints are feasible. By Claim 1, there exists a positive constant λ such that $(z_1^{opt})^T z_1^{opt} \geq \lambda (z_2^{opt})^T z_2^{opt}$. This together with the fact $(x^{opt})^T x^{opt} = (z_1^{opt})^T z_1 + (z_2^{opt})^T z_2$ (since V is unitary) gives

$$|z_1^{opt}|_2^2 \geq \frac{1}{1 + \frac{1}{\lambda}} |x^{opt}|_2^2$$

Thus, $|Ex^{opt}|_2^2 = |U_E \Sigma z_1^{opt}|_2^2 = \left| \begin{bmatrix} U_E \\ U_G \end{bmatrix} \Sigma z_1^{opt} \right|_2^2 = |\Sigma z_1^{opt}|_2^2 \geq \bar{\lambda} |x^{opt}|_2^2$ where $\bar{\lambda} = \underline{\sigma}(\Sigma)\frac{1}{1+\frac{1}{\lambda}}$ and $\underline{\sigma}(\Sigma) > 0$ is the smallest singular value of Σ.

$$
\begin{aligned}
J &= \min_{x \in X} (a_0 + Ex)^T W (a_0 + Ex) \\
&= \min_{x \in X} (-a_0 - Ex)^T W (-a_0 - Ex) \\
&\leq (-a_0)^T W (-a_0) - (Ex^{opt})^T W (Ex^{opt}) \qquad \text{(by Claim 3)} \\
&= a_0^T W a_0 - (Ex^{opt})^T W (Ex^{opt}) \\
&\leq a_0^T W a_0 - \underline{\sigma}(W)(Ex^{opt})^T (Ex^{opt}) \\
&\leq a_0^T W a_0 - \gamma (x^{opt})^T x^{opt}
\end{aligned}
$$

where $\gamma = \underline{\sigma}(W)\bar{\lambda}$ and $\underline{\sigma}(W) > 0$ is the smallest singular value of W. □

Remark 20 *As one can see, the optimal solution of* $J = \min_{x \in X}(a_0 + Ex)^T W (a_0 + Ex)$ *may not be unique. If we do not determine the unique optimal solution as* $\arg\min x^T x$ *over all feasible solutions for which* J *has the optimal value, then this claim does not hold in general.*

Now we are ready to prove Theorem 14.

Proof. WLOG, assume that $u^{min} + \delta \leq -w(k) \leq u^{\max} - \delta \ \forall \ k \geq 0$, where $\delta > 0$ is constant, and $|w(k+1) - w(k)| \leq \epsilon \ \forall \ k \geq 0$.[5] The future disturbance is estimated by assuming that it is step-like, *i.e.* $\hat{w}(k+i|k) = w(k-1) \ \forall \ i \geq 0$ where $\hat{w}$ denotes the estimate of w. Thus $u(k+N-1|k) + \hat{w}(k+N-1|k) = 0$ is always feasible, *i.e.* $\Phi(k, n_{\max}) = 0 \ \forall \ k \geq 0$ is always feasible. Only $N-1$ control moves are used to minimize the objective function. Let $Q(j|i)$ be the coefficients calculated at time j with reference time at i, *i.e.* i is treated as the initial time (0). We have

[5] By assumptions on the disturbance, this is always possible by appropriately defining the initial time.

$$\begin{bmatrix} P(-n+1) \\ P(-n+2) \\ \vdots \\ P(0) \end{bmatrix} Q(k|k+N+n_b-1) = \begin{bmatrix} y(k+N-n+n_b-1|k) \\ \vdots \\ y(k+N+n_b-2|k) \\ y(k+N+n_b-1|k) \end{bmatrix} =$$

$$C \begin{bmatrix} y(k) \\ \vdots \\ y(k-n+1) \end{bmatrix} + D \begin{bmatrix} u(k+N-n+2-n_b) - w(k+N-n+2-n_b) \\ \vdots \\ u(k|k) - w(k-1) \\ u(k+1|k) - w(k-1) \\ \vdots \\ u(k+N-2|k) - w(k-1) \\ u(k+N-1|k) - w(k-1) = 0 \end{bmatrix} \quad (4.22)$$

$$\begin{bmatrix} P(-n+1) \\ P(-n+2) \\ \vdots \\ P(0) \end{bmatrix} Q(k+1|k+N+n_b-1) = \begin{bmatrix} y(k+N-n+n_b-1|k+1) \\ \vdots \\ y(k+N+n_b-2|k+1) \\ y(k+N+n_b-1|k+1) \end{bmatrix} =$$

$$C \begin{bmatrix} y(k) \\ \vdots \\ y(k-n+1) \end{bmatrix} + D \begin{bmatrix} u(k+N-n+2-n_b) - w(k+N-n+2-n_b) \\ \vdots \\ u(k) - w(k) \\ u(k+1|k+1) - w(k) \\ \vdots \\ u(k+N-1|k+1) - w(k) \end{bmatrix} \tag{4.23}$$

Subtraction of the above two equations and a few lines of algebra give

$$Q(k+1|k+N+n_b-1) = Q(k|k+N+n_b-1) + F\Delta v_{k+1} + G(w(k) - w(k-1)) \tag{4.24}$$

where $\Delta v_{k+1} = [u(k+1|k+1) \ \cdots \ u(k+N-1|k+1)]^T - [u(k+1|k) \ \cdots \ u(k+N-1|k)]^T$ and F and G are defined in an obvious manner. Or equivalently, for $\alpha = 1, \cdots, n_{\max}$, we have

$$Q_\alpha(k+1|k+N+n_b-1) = Q_\alpha(k|k+N+n_b-1) + F_\alpha \Delta v_{k+1} + G_\alpha(w(k) - w(k-1)) \tag{4.25}$$

where $F = \begin{bmatrix} F_1 \\ \vdots \\ F_{n_{\max}} \end{bmatrix}$ and G is defined similarly.

Remark 21 *Notice that $Q(k+1|k+N+2)$ may not be necessarily equal to $Q(k+1|k+N+1)$. However, by Corollary 1, $Q_i(k+1|k+N+2) = 0 \ \forall \ i \geq \alpha$ and $Q_\alpha(k+1|k+N+2)^T W_\alpha Q_\alpha(k+1|k+N+2) = Q_\alpha(k+1|k+N+1)^T W_\alpha Q_\alpha(k+1|k+N+1)$ if and only if $Q_i(k+1|k+N+1) = 0 \ \forall \ i \geq \alpha$,*

The optimization problem, with slight abuse of notations, becomes the following:

$$J = \min_{\Delta v_{k+1}} \left[Q_\alpha + F_\alpha \Delta v_{k+1}\right]^T W_\alpha \left[Q_\alpha + F_\alpha \Delta v_{k+1}\right] \tag{4.26}$$

subject to

$$\begin{cases} F_{\alpha+i}\Delta v_{k+1} = G_\alpha(w(k) - w(k-1)) = O(\epsilon) \;\; \forall\, i = 1, \cdots, n_{\max} - \alpha \\ u(k+N|k+1) - u(k+N-1|k) = -w(k) + w(k-1) \\ u^{min} \le u(k+i|k+1) \le u^{\max} \;\forall\, i = 1, \cdots, m \end{cases} \tag{4.27}$$

The following claim is obvious.

Claim 5 *The matrix consisting of the last n columns of F is nonsingular if $N \geq n+1$.*

Proof. Since the system is controllable, we can transfer any initial state to an arbitrary state with at most n control moves if the controls are unconstrained. Since the last control move is such that $u(k+N-1|k) + w(k-1) = 0$, we can take the coefficients from any initial condition to any arbitrary values with $n+1$ control moves. Therefore, the matrix consisting of the last n columns of F must be nonsingular if $N \geq n+1$. □

The proof is completed with the following two claims.

Claim 6 *If $w(k) - w(k-1) = 0 \;\forall\, k \geq 1$, then*

$$J(k+n+1, \alpha) \leq \max(J(k,\alpha) - \eta(\alpha), 0) \;\forall\; \alpha \geq 1$$

where $\eta(\alpha)$ is a positive constant that depends on α if $N \geq n+1$ and only if $N \geq n - n_{\text{modes}} + 2$.

Proof. ($\Rightarrow$) $N \geq n+1$.
Case 1: Suppose $|\Delta v_{k+i}|_\infty \leq \beta$, $\forall\, i = 1, \cdots, n$ and let $\beta = \frac{\min(|u^{min} - w(k)|, |u^{\max} - w(k)|)}{n+1} \geq \frac{\delta}{n+1} > 0$. We have $|u(k+N+i|k+n) - u(k+N+i|k)| \leq \beta n \;\forall\, i = -1, \cdots, n-1$. Since $u(k+N+i|k) = u(k+N-1|k) = -w(k-1) \;\forall\, i \geq 0$, $|u(k+N+i|k+n) + w(k-1)| \leq \beta n = \frac{n}{n+1}\delta \;\forall\, i \geq -1$. This together with the fact $u^{min} + \delta \leq -w(k-1) \leq u^{\max} - \delta$ gives $\min(u(k+N+i|k+n) - u^{min}, u^{\max} - u(k+N+i|k+n)) \geq \beta \;\forall\, i = -1, \cdots, n-1$. Thus at the sampling time $k+n+1$, the last $n+1$ elements of Δv_{k+n+1}, denoted by $\Delta v'$, can be varied within $\pm\beta$, *i.e.* $-\beta \geq v^{min} \leq \Delta v' \leq v^{\max} \geq \beta$. Assume that the

first $N-n-1$ elements of Δv_{k+n+1} are zeros. Then we have

$$J(k+n+1,\alpha) \leq \min_{\Delta v'} \left[Q_\alpha + H_1 \Delta v'\right]^T W_\alpha \left[Q_\alpha + H_1 \Delta v'\right]$$

subject to

$$\begin{cases} H_2 \Delta v' = 0 \\ -\beta \geq v^{min} \leq \Delta v' \leq v^{\max} \geq \beta \end{cases}$$

where $H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$ is the last n columns of $\begin{bmatrix} F_{\alpha+1} \\ \vdots \\ F_{n_{\max}} \end{bmatrix}$. Notice that the inequality follows from the assumption that the first $N-n-1$ elements of Δv_{k+n+1} are zeros. By Claim 5, H must have full row rank. Then there exists a positive constant (it can be taken, for example, as the largest radius of balls centered at the origin within the set) $\eta(\alpha)$ such that $J(k+n+1,\alpha) \leq \max(J(k,\alpha)-\eta(\alpha),0)$.

Case 2: If $|\Delta v_{k+i}|_\infty \geq \beta$ for some $i \in \{1,\cdots,n\}$, then by Claim 4, $J(k+n,\alpha) \leq J(k,\alpha) - \gamma\beta^2$. This completes the proof for the *if* part.

($\Leftarrow$) If $N \leq n - n_{\text{modes}} + 2$, then for $\alpha = 1$, $\begin{bmatrix} F_2 \\ \vdots \\ F_{n_{\max}} \end{bmatrix}$ has more columns than rows and the only solution, if feasible, is $\Delta v_{k+1} = 0$ for some initial conditions. Thus no degree of freedom is left to minimize $J(k,1)$. For some initial conditions, $J(k,1)$ cannot be reduced to zero. □

Claim 7 *For sufficiently large k, there exists an integer o, $2(n+1) \geq o \geq n+1$ such that*

$$J(k+o,\alpha) \leq \max(J(k,\alpha)-\eta'(\alpha),0) \;\forall\; \alpha \geq 1$$

where $\eta'(\alpha) > 0$ if $N \geq n+1$.

Proof. Because of the disturbance, the constraints (4.27) may not be feasible at the sampling time $k+1$ even though they are feasible at the sampling time k. We want to show, however, that for sufficiently large k, or equivalently for sufficiently small ϵ, there exists an integer $1 \leq l \leq n+1$ such that the constraints are feasible at the sampling time $k+l$. Suppose that the constraints are not feasible for all $l \leq n$; otherwise, we are done. By Claim 4, $\Delta v_{k+i} \sim O(\epsilon)\ \forall\, i = 1, \cdots, n$. Since there exists a positive constant δ such that $u^{min} + \delta \leq -w(k+i) \leq u^{\max} - \delta\ \forall\, i \geq 0$, for sufficiently small ϵ, following the similar arguments as in the proof of Claim 6, the last $n+1$ elements of Δv_{k+n+1}, denoted by $\Delta v'$, are allowed to vary within $\pm\beta$ where $\beta > 0$ is as defined in the proof of Claim 6, *i.e.* $-\beta \geq x^{min} \leq \Delta v' \leq x^{\max} \geq \beta$. Thus $\begin{bmatrix} F_{\alpha+1} \\ \cdots \\ F_{n_{\max}} \end{bmatrix}$ subject to the constraints $-\beta \geq x^{min} \leq \Delta v' \leq x^{\max} \geq \beta$ covers a ball centered at the origin with radius of ρ. For sufficiently small ϵ, $\begin{bmatrix} F_{\alpha+1} \\ \cdots \\ F_{n_{\max}} \end{bmatrix} = O(\epsilon)$ must be feasible. Therefore, for sufficiently small ϵ, there exists an integer $1 \leq l \leq n+1$ such that the constraints are feasible at the sampling time $k+l$.

Suppose that at the sampling time o, where $2(n+1) \geq o \geq n+1$, the constraints are feasible. By Claim 4, the control moves in making the constraints feasible are $\mathcal{O}(\epsilon)$. Therefore, the effect of the control moves on $J(k+o, \alpha)$ is $O(\epsilon)$. This combined with the previous claim gives

$$J(k+o, \alpha) \leq \max(J(k, \alpha) - \eta(\alpha) + \mathcal{O}(\epsilon), 0)$$

Thus for sufficiently small ϵ, we have

$$J(k+o, \alpha) \leq \max(J(k, \alpha) - \eta'(\alpha), 0)$$

where $\eta'(\alpha) = \eta(\alpha) - O(\epsilon) > 0$. $\square$

Thus, $J(k, 0) \to 0$ as $k \to \infty$ which in turn yields $y(k) \to r$ asymptotically. This completes the proof of Theorem 14. $\square$

Chapter 5 Infinite Horizon MPC with Mixed Constraints—Unstable Systems

Summary

In this chapter, we analyze and characterize the domain of attractability for a linear unstable discrete-time system with bounded controls. An algorithm is proposed to construct the domain of attractability. We show that the Infinite Horizon MPC with Mixed Constraints algorithm generates a class of (nonlinear) control laws that stabilize the system for all initial conditions in the domain of attractability.

5.1 Introduction

It is well known [59, 84] that a linear time-invariant discrete-time system is *globally* stabilizable with bounded controls if and only if it is stabilizable and all the eigenvalues are inside the closed unit disk. In Chapters 3 and 4, we have shown that the Infinite Horizon MPC with Mixed Constraints (IHMPCMC) algorithm (with the input horizon chosen properly) automatically generates a class of (nonlinear) control laws that globally stabilize any system for which global stabilization is possible. Since global stabilization is *not* possible for systems with poles outside the unit disk, it may be desirable, to characterize and determine the domain of attractability (referred to as the *maximum region of recoverability* in [59]), i.e. the set of *all* initial conditions for which a stabilizing control law exists, but very little work has been done.

Most of the work in the literature (see, for example, [38, 33, 5]), *not* necessarily applicable to unstable systems, has been to determine an invariant set for a *linear* controller. A set is said to be invariant if the state remains in the set for every initial condition started in the set. Disturbances can also be taken into account to construct such an invariant set [5]. In general, however, such an invariant set is a *conservative* approximation of the domain of attractability. This is because that only *linear* control laws are allowed. Also in many cases, the control law is constructed such that the control input does not saturate.

In this chapter, we analyze and characterize the domain of attractability for a linear unstable discrete-time system with hard input constraints and soft output constraints. An algorithm is proposed to determine the domain of attractability within an *arbitrary* accuracy. We show that the IHMPCMC algorithm generates a class of (nonlinear) control laws that stabilize the system for all initial conditions in the domain of attractability.

This chapter is organized as follows: In Section 5.2, the domain of attractability is analyzed and determined. We show in Section 5.3 that the Infinite Horizon MPC with Mixed Constraints algorithm generates a class of (nonlinear) stabilizing control laws. Several examples are presented in Section 5.4. Section 5.5 concludes the chapter.

5.2 Domain of Attractability

Consider the following linear time invariant discrete-time system,

$$x(k+1) = Ax(k) + Bu(k), |u(k)|_\infty \leq 1, k \geq 0 \tag{5.1}$$

where $x(k) \in \Re^{n_x}, u(k) \in \Re^{n_u}$, and A and B are matrices of appropriate dimensions. The domain of attractability, W, is defined as follows.

Definition 9 *The domain of attractability, denoted by W, is the set of* all *initial conditions for which there exists a sequence of controls $\{u(0), u(1), \cdots, u(K), 0, 0, \cdots\}$, $|u(i)|_\infty \leq 1 \; \forall \, i \geq 0$, for some finite integer K such that the state approaches the origin asymptotically.*

Remark 22 It is without loss of generality (WLOG) to assume that $|u|_\infty \leq 1$ in (5.1) instead of $u^{min} \leq u \leq u^{max}$. Let P be a diagonal matrix whose diagonal elements equal $\frac{1}{2}(u^{max} - u^{min})$. By defining $u = P\tilde{u} + \frac{1}{2}(u^{max} + u^{min})$ and $x = \tilde{x} + \frac{1}{2}(I-A)^{-1}B(u^{max} + u^{min})$,[1] we can transform (5.1) with $u^{min} \leq u \leq u^{max}$ into

[1] Here we assume that A does not have eigenvalues at 1.

$\tilde{x}(k+1) = A\tilde{x}(k) + \tilde{B}\tilde{u}(k)$ where $|\tilde{u}|_\infty \leq 1$ and $\tilde{B} = BP$.

The following result is immediate from Definition 9.

Theorem 15 *There exists a control law such that the closed loop system is asymptotically stable if and only if the initial condition $x(0) \in W$.*

For stabilizable systems with $\rho(A) \leq 1$, where $\rho(A)$ denotes the spectral radius of A, Sontag proved that W is $\Re^{n_x}$.

Theorem 16 (Sontag 1984 [84]) $W = \Re^{n_x}$ *if and only if (A, B) is stabilizable and $\rho(A) \leq 1$.*

Assume, WLOG, the system (5.1) is represented as follows.

$$\begin{bmatrix} x_s(k+1) \\ x_c(k+1) \\ x_u(k+1) \end{bmatrix} = \begin{bmatrix} A_s & 0 & 0 \\ 0 & A_c & 0 \\ 0 & 0 & A_u \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_c(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} B_s \\ B_c \\ B_u \end{bmatrix} u(k) \qquad (5.2)$$

where $A_s \in \Re^{n_{x_s} \times n_{x_s}}$ has all eigenvalues inside the unit circle, $A_c \in \Re^{n_{x_c} \times n_{x_c}}$ on the unit circle, and $A_u \in \Re^{n_{x_u} \times n_{x_u}}$ outside the unit circle. By Theorem 16, the domain of attractability for the system without any poles outside the unit circle is $\Re^{n_{x_s+x_c}}$. The following corollary states that the poles outside the unit circle do not change that.

Corollary 6 [2] *Consider the system described by (5.2) and assume that $\{A, B\}$ is stabilizable. The domain of attractability for x_s and x_c are $\Re^{n_{x_s}}$ and $\Re^{n_{x_c}}$, respectively.*

Proof. It is obvious that the domain of attractability for x_s is $\Re^{n_{x_s}}$: no control action is necessary to stabilize *any* initial condition $x_s(0) \in \Re^{n_{x_s}}$. So we only have to show that the domain of attractability for x_c is $\Re^{n_{x_c}}$. WLOG, assume that A has all eigenvalues on and/or outside the unit circle. Then stabilizability and controllability of $\{A, B\}$ are equivalent.

[2] See an earlier proof by LeMay [59] for continuous-time systems.

Denote the domain of attractability for x_u by W^u. Let $x_u(0) \in W^u$. Then there exists a finite integer K such that $x_u(K+1) = 0$. So it is WLOG to assume that $x_u(0) = 0$ and to show that the domain of attractability for x_c is $\Re^{n_{xc}}$, i.e. $[x_c(k) \;\; x_u(k)]^T \to 0$ as $k \to \infty$ for all $x_c(0) \in \Re^{n_{xc}}$ and $x_u(0) = 0$. Let α be some integer. We have

$$\begin{bmatrix} x_c(k+\alpha) \\ x_u(k+\alpha) \end{bmatrix} = \begin{bmatrix} A_c^\alpha & 0 \\ 0 & A_u^\alpha \end{bmatrix} \begin{bmatrix} x_c(k) \\ x_u(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix} v(k)$$

where

$$\begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix} = \begin{bmatrix} A_c^{\alpha-1}B_c & \cdots & B_c \\ A_u^{\alpha-1}B_u & \cdots & B_u \end{bmatrix} \qquad v(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k+\alpha-1) \end{bmatrix}$$

Since the system is controllable, α exists such that $\begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix}$ has full row rank. Consider a linear feedback control law $v(k) = Fx_c(k)$. We have

$$\begin{bmatrix} x_c(k+\alpha) \\ x_u(k+\alpha) \end{bmatrix} = \begin{bmatrix} A_c^\alpha + \tilde{B}_c F & 0 \\ \tilde{B}_u F & A_u^\alpha \end{bmatrix} \begin{bmatrix} x_c(k) \\ x_u(k) \end{bmatrix}$$

Given any $x_c(0)$ $\Re^{n_{xc}}$, if F exists such that

$$\begin{aligned} \tilde{B}_u F &= 0 \\ \rho(A_c^\alpha + \tilde{B}_c F) &< 1 \\ |Fx_c(k)| &\leq 1 \;\forall\; k \geq 1 \end{aligned}$$

then it follows that the domain of attractability x_c is $\Re^{n_{xc}}$.

WLOG, assume that $\begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix}$ is square and nonsingular: just set some rows of F to zeros if $\begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix}$ is non-square. Let $\tilde{B}_u^{\perp}$ be the orthogonal complement of $\tilde{B}_u$, i.e. $\begin{bmatrix} \tilde{B}_u^{\perp} \\ \tilde{B}_u \end{bmatrix}$ is square and nonsingular and $\tilde{B}_u^{\perp}\tilde{B}_u^T = 0$. Let $\tilde{B}_c = C_1\tilde{B}_u^{\perp} + C_2\tilde{B}_u$ and $F = (\tilde{B}_u^{\perp})^T E$. Thus $\tilde{B}_u F = 0 \ \forall \ E$. Clearly C_1 and $\tilde{B}_u^{\perp}(\tilde{B}_u^{\perp})^T$ are nonsingular[3] which implies that E exists such that $\rho(A_c^{\alpha} + \tilde{B}_c F) = \rho(A_c^{\alpha} + C_1\tilde{B}_u^{\perp}(\tilde{B}_u^{\perp})^T E) < 1$. From the results by Lin and Saberi [61], E exists such that $|Fx_c(k)|_{\infty} \leq 1 \ \forall \ k$ for any initial condition $x_c(0) \in \Re^{n_{x_c}}$. Therefore, the domain of attractability for x_c is $\Re^{n_{x_c}}$. □

Therefore, we *only* need to determine W^u, the domain of attractability for x_u. For the *rest of this section*, unless specified otherwise, we assume, WLOG, that A has *all* the eigenvalues outside the unit circle, i.e. $A = A_u$ and $x = x_u$. The state at time k can be written as

$$x(k) = A^k x(0) + [A^{k-1}B \quad \cdots \quad B] \begin{bmatrix} u(0) \\ \vdots \\ u(k-1) \end{bmatrix} \tag{5.3}$$

Let W_N^u be the set of all initial conditions for which there exists a sequence of controls $\{u(0), u(1), \cdots, u(N-1)\}, |u(i)|_{\infty} \leq 1 \ \forall \ i \geq 0$ such that $x(N) = 0$. Thus, $W^u = \lim_{N\to\infty} W_N^u$. W_N^u can be written as follows.

$$W_N^u = \left\{ z : z = [A^{-1}B \quad \cdots \quad A^{-N}B] \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix}, |u(i)|_{\infty} \leq 1, i \geq 0 \right\} \tag{5.4}$$

---

[3]That C_1 is nonsingular can be seen as follows: $\begin{bmatrix} \tilde{B}_c \\ \tilde{B}_u \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{B}_u^{\perp} \\ \tilde{B}_u \end{bmatrix}$.

Some properties of W_N^u and W^u are stated here.

Lemma 3 *W_N^u and W^u are bounded, convex, and symmetric.*

Proof. Since A contains poles strictly outside the unit circle, $\rho(A^{-1}) < 1$. We have

$$||A^{-i}||_\infty \leq c\gamma^i i^\beta, \gamma \in [0,1) \text{ for some integer } \beta \text{ and some positive constant } c$$

Suppose $x(0) \in W_N^u$. We have

$$\begin{aligned} |x(0)|_\infty &= \left|\sum_{i=1}^{N} A^{-i}Bu(i)\right|_\infty \\ &\leq \sum_{i=1}^{N} |A^{-i}Bu(i)|_\infty \\ &\leq c_1 \sum_{i=1}^{N} ||A^{-i}||_\infty \\ &\leq c_1 c \sum_{i=1}^{N} \gamma^i i^\beta \\ &< \infty \;\forall\; N \end{aligned}$$

where $c_1 = |B|_\infty$. Thus, W_N^u is bounded. The convexity of W_N^u follows by observing that convexity is preserved for linear transformations. For $x(0) \in W_N^u$, there exists a sequence of controls $\{u(0), \cdots, u(N-1)\}, |u(i)|_\infty \leq 1 \;\forall\; i$ such that $x(0) = \sum_{i=1}^N A^{-i}Bu(i-1)$. Clearly, $-x(0) = \sum_{i=1}^N A^{-i}B(-u(i-1))$ must also belong to W_N^u. Therefore, W_N^u is symmetric. The proof for W^u follows by replacing N with ∞. □

Remark 23 *Although W_N^u is* closed, *W^u is* open.

In the next few subsections, we discuss several ways to characterize W^u and therefore W.

5.2.1 Exact Characterization of W_N^u

In this section, we propose an algorithm to determine W_N^u. Let us first present some preliminary results.

Lemma 4 *Consider the following sets.*

$$X_1 = \{z : H_1 z \leq h_1\}$$

$$X_2 = \{z : H_2 z \leq h_2\}$$

Assume that both X_1 and X_2 are bounded. Denote the vertices of X_1 by $\mu_i, i = 1, \cdots, n_1$, and the vertices of X_2 by $\nu_i, i = 1, \cdots, n_2$. Let

$$X = \{z : z = z_1 + z_2, z_1 \in X_1, z_2 \in X_2\}$$

Then X is bounded and is the smallest convex set which contains the points $\mu_i + \nu_j, i = 1, \cdots, n_1, j = 1, \cdots, n_2$. Furthermore, X can be represented as follows:

$$X = \{z : Hz \leq h\}$$

Proof. The convexity of X can be shown as follows: Suppose $y, z \in X, y_1, z_1 \in X_1, y_2, z_2 \in X_2$ and $0 \leq \lambda \leq 1$. $\lambda y + (1-\lambda)z = \lambda(y_1 + y_2) + (1-\lambda)(z_1 + z_2) = (\lambda y_1 + (1-\lambda)z_1) + (\lambda y_2 + (1-\lambda)z_2) \in X$ since X_1 and X_2 are convex. X is bounded since X_1 and X_2 are bounded.

Next we want to prove the following: If X contains the points $\mu_i + \nu_j, i = 1, \cdots, n_1, j = 1, \cdots, n_2$, then $y_1 + y_2 \in X \ \forall \ y_1 \in X_1, y_2 \in X_2$. By convexity of X, for $0 \leq \lambda_1 \leq 1$, we have $\lambda_1(\mu_i + \nu_{j_1}) + (1-\lambda_1)(\mu_i + \nu_{j_2}) \in X \ \forall \ i, j_1, j_2$ which yields

$$\mu_i + \lambda_1 \nu_{j_1} + (1-\lambda_1)\nu_{j_2} \in X \ \forall \ i, j_1, j_2$$

Similarly, for $0 \leq \lambda_2 \leq 1$, we have

$$\lambda_2(\mu_{i_1} + \lambda_1 \nu_{j_1} + (1-\lambda_1)\nu_{j_2}) + (1-\lambda_2)(\mu_{i_2} + \lambda_1 \nu_{j_1} + (1-\lambda_1)\nu_{j_2}) \in X$$

which yields

$$(\lambda_2\mu_{i_1} + (1-\lambda_2)\mu_{i_2}) + (\lambda_1\nu_{i_1} + (1-\lambda_1)\nu_{i_2}) \in X \ i_1, i_2 = 1, \cdots, n_1, j_1, j_2 = 1, \cdots, n_2$$

Thus, all points which are sum of the points on edges of X_1 and X_2 belong to X. By similar arguments, one can show easily that $y_1 + y_2 \in X \ \forall \ y_1 \in X_1, y_2 \in X_2$. Clearly the smallest convex set which contains a finite number of points is a polytope. □

Recall

$$\begin{aligned} W_N^u &= \left\{ x(0) : x(0) = \sum_{i=1}^{N} A^{-i}Bu(i), |u(i)|_\infty \le 1, i \ge 0 \right\} \\ &= \left\{ x(0) : x(0) = \sum_{i=1}^{N} x_i, x_i \in X_i \right\} \end{aligned}$$

where

$$X_i = \{z : z = A^{-i}By, |y|_\infty \le 1\}$$

W_N^u can then be determined via the following algorithm.

Algorithm 1 *Data: A, B, and N. Denote the set of vertices of the polytope X_i by $V(X_i)$.*

Step 0 *Set $i = 1$. Determine $V(X_1)$ and set $V(X) = V(X_1)$.*

Step 1 *If $i = N$, go to* **Step 2**. *Otherwise, set $i = i+1$. Determine $V(X_i)$. Calculate $PV(X) = \{\mu : \mu = y + z, y \in V(X), z \in V(X_i)\}$. Eliminate all points from $PV(X)$ that are not vertices for the smallest polytope that covers all points in $PV(X)$.[4] Set $V(X) = PV(X)$. Go to* **Step 1**.

Step 2 *Construct the polytope with vertices $V(X)$.*

Let $PV(X) = \{\mu_1, \cdots, \mu_M\}$. We can determine if a point in $PV(X)$, say μ_i, is a vertex by solving the following optimization problem, which can be cast as a linear

[4]Since W_N^u is symmetric, we only have to check half of total number of vortices.

program.

$$J = \min_{\alpha} \left| \mu_i - \sum_{j=1}^{M} \delta_j \mu_j \right|_{\infty}$$

$$\text{subject to} \quad \delta_j \geq 0 \quad \forall\, j, \delta_i = 0, \sum_{j=1}^{M} \delta_j = 1$$

It is clear that μ_i is a vortex if and only if $J > 0$.

Remark 24 *Constructing W_N^u this way requires to repeat* **Step 1** $N-1$ *times, i.e. $N-1$ operations in set addition. Since doing set addition may be computationally expensive, we can reduce the number of set addition as follows: Define D_i of full row rank and $l(N) < N$* [5] *such that*

$$[D_1 \ \cdots \ D_{l(N)}] = [A^{-1}B \quad \cdots \quad A^{-N}B]$$

Then W_N^u can be rewritten as

$$W_N^u = \left\{ x(0) : x(0) = \sum_{i=1}^{l(N)} D_i v(i), |v(i)|_{\infty} \leq 1, i \geq 0 \right\}$$

By defining X_i similarly, we only have to repeat **Step 1** $l(N)-1$ *times, i.e. $l(N)-1$ operations of set addition. Of course, in this case, it may take more computational time to determine the vertices of X_i.*

5.2.2 Subsets of W^u

Let $C = [A^{-1}B \quad \cdots \quad A^{-n}B]$, where n is the smallest integer such that C has full row rank.[6] We have

$$W^u = \left\{ x(0) : x(0) = \sum_{i=0}^{\infty} (A^{-n})^i C U_n(i), |U_n(i)|_{\infty} \leq 1 \right\}$$

[5] D_i's and $l(N)$ are clearly not unique.

[6] Since (A, B) is controllable, such an n exists.

where $U_n(i) = [u(i \cdot n) \quad \cdots \quad u((i+1)n-1)]^T$. Let the set W^u_{in} be generated by assuming $U_n(i) = U_n(0)\ \forall\ i \geq 1$, i.e.

$$\begin{aligned} W^u_{in} &= \left\{x(0) : x(0) = \sum_{i=0}^{\infty}(A^{-n})^i C U_n(0), |U_n(0)|_\infty \leq 1\right\} \\ &= \left\{x(0) : x(0) = (I - A^{-n})^{-1} C U_n(0), |U_n(0)|_\infty < 1\right\} \end{aligned}$$

Then we must have $W^u_{in} \subseteq W^u$. If C is square and nonsingular,[7] then

$$W^u_{in} = \left\{x(0) : |C^{-1}(I - A^{-n})x(0)|_\infty < 1\right\} \tag{5.5}$$

5.2.3 Supersets of W^u

From $x(0) = \sum_{i=0}^{\infty}(A^{-n})^i C U_n(i)$, we have

$$|Tx(0)|_\infty \leq \sum_{i=0}^{\infty} |T(A^{-n})^i C U_n(i)|_\infty \leq \sum_{i=0}^{\infty} |T(A^{-n})^i C|_\infty$$

where T is some nonsingular weighting matrix. Thus, a superset of W^u, W^u_{out}, can be defined as follows:

$$W^u_{out} = \left\{x(0) : |Tx(0)|_\infty \leq \sum_{i=0}^{\infty} |T(A^{-n})^i C|_\infty\right\} \supseteq W^u$$

5.2.4 Characterization of W^u

W^u_N can be characterized exactly and can be used to approximate W^u. Since A has poles outside the unit circle, A^{-N} approaches zero as $N \to \infty$. We can approximate W^u with an arbitrary accuracy. Then the techniques presented in Sections 2.2 and 2.3 can be used to bound the approximation error, i.e. we have the following relations:

$$W^u \supseteq \{x(0) : x(0) = y + z, y \in W^u_N, z \in W^u_{in_N}\} \tag{5.6}$$

$$W^u \subseteq \{x(0) : x(0) = y + z, y \in W^u_N, z \in W^u_{out_N}\} \tag{5.7}$$

[7] For single input controllable systems, C is always nonsingular if C is square.

where

$$
\begin{aligned}
W^u_{in_N} &= \left\{x(0) : x(0) = A^{-N}(I - A^{-n})^{-1}CU_n(0), |U_n(0)|_\infty < 1\right\} \\
W^u_{out_N} &= \left\{x(0) : |Tx(0)|_\infty \leq \sum_{i=0}^{\infty} |TA^{-N}(A^{-n})^i C|_\infty\right\}
\end{aligned}
$$

5.2.5 Characterization of W

Once we have determined W^u, W for system (5.2), i.e. A now has poles inside, on, and outside the unit circle, is determined as well. It is simply given as

$$
W = \left\{ x(0) : x(0) = \begin{bmatrix} x_s(0) \\ x_c(0) \\ x_u(0) \end{bmatrix}, x_s(0) \in \Re^{n_s}, x_c(0) \in \Re^{n_c}, x_u(0) \in W^u \right\}
$$

It can be approximated by W_N which is given as follows:

$$
W_N = \left\{ x(0) : x(0) = \begin{bmatrix} x_s(0) \\ x_c(0) \\ x_u(0) \end{bmatrix}, x_s(0) \in \Re^{n_s}, x_c(0) \in \Re^{n_c}, x_u(0) \in W^u_N \right\}
$$

5.3 Stabilizing Control Laws

For any initial condition $x(0) \in W$, Theorem 15 states the existence of a stabilizing control law. In this section, we give a necessary and sufficient condition for the IHMPCMC algorithm to be stabilizing.

Define the objective function as

$$
\begin{aligned}
\Phi_k = \sum_{i=1}^{\infty} x(k+i|k)^T \Gamma_x x(k+i|k) \quad &+ \sum_{i=0}^{H_c} \Big[u(k+i|k)^T \Gamma_u u(k+i|k) + \\
&\Delta u(k+i|k)^T \Gamma_{\Delta u} \Delta u(k+i|k) \Big]
\end{aligned}
\tag{5.8}
$$

where $\Gamma_x > 0, \Gamma_u > 0, \Gamma_{\Delta u} \geq 0, \Delta u(k+i|k) = u(k+i|k) - u(k+i-1|k)$, and H_c is finite. Γ_x, Γ_u, and $\Gamma_{\Delta u}$ are symmetric. $(\cdot)(k+i|k)$ denotes the variable $(\cdot)$ at sampling time $k+i$ predicted at sampling time k and $(\cdot)(k) = (\cdot)(k|k)$.

Define *Controller IHMPCMC* as follows.

Definition 1 Controller IHMPCMC: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \quad \Phi_k + \epsilon(k)^T \Gamma_\epsilon \epsilon(k)$$

$$\text{subject to} \quad \begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, \cdots, H_c - 1 \\ u(k+i|k) = 0 & i = H_c, \cdots, \infty \\ x(k+i|k) \leq \mathcal{X}_{\epsilon(k)} & i = 0, 1, \cdots, \infty \\ \epsilon(k) \geq 0 & \end{cases} \tag{5.9}$$

where $\Gamma_\epsilon > 0$ is diagonal, Φ_k is defined by (5.8), and $\mathcal{U}$ and $\mathcal{X}$ are given as

$$\begin{aligned} \mathcal{U} &\triangleq \{u : |u| \leq 1\} \\ \mathcal{X}_\epsilon &\triangleq \left\{ x : [F_x \quad F_u] \begin{bmatrix} x \\ u \end{bmatrix} \leq f + \epsilon, \epsilon \geq 0, u \in \mathcal{U} \right\} \end{aligned}$$

Remark 25 *Follow similar arguments leading to Theorem 8 in Chapter 3, we can replace $x(k+i|k) \leq \mathcal{X}_{\epsilon(k)}, i = 0, 1, \cdots, \infty$, by $x(k+i|k) \leq \mathcal{X}_{\epsilon(k)}, i = 0, 1, \cdots, M$, where M is finite.*

When A has no poles on the unit circle and if $x(0) \in W_N$, then the optimizing problem (5.9) is feasible for all $H_c \geq N$. Thus we have the following result.

Theorem 17 *Assume that A has no eigenvalues on the unit circle. Then* Controller IHMPCMC *is stabilizing for all $x(0) \in W_N$ if and only if $H_c \geq N$.*

Proof. If $x(0) \in W_N$, then by definition there exists a sequence of controls $\{u(0), \cdots,$

$u(N-1)\}$ such that $x_u(N) = 0$ and $x_s(k) \to 0$ exponentially as $k \to \infty$. The optimization problem (5.9) has a feasible solution for *all* $x(0) \in W_N$ *if and only if* $H_c \geq N$. Then J_0 is bounded. At sampling time $k+1$, the control sequence of $\{u(k+1|k), \cdots, u(k+H_c-1|k), 0\}$ results in a finite objective that equals

$$J_k - \left[x(k)^T \Gamma_x x(k) + u(k)^T \Gamma_u u(k) + \Delta u(k)^T \Gamma_{\Delta u} \Delta u(k)\right]$$

Thus, we have

$$J_{k+1} \leq J_k - \left[x(k)^T \Gamma_x x(k) + u(k)^T \Gamma_u u(k) + \Delta u(k)^T \Gamma_{\Delta u} \Delta u(k)\right]$$

which yields

$$J_{k+1} + \sum_{i=0}^{k+1} \left[x(i)^T \Gamma_x x(i) + u(i)^T \Gamma_u u(i) + \Delta u(i)^T \Gamma_{\Delta u} \Delta u(i)\right] \leq J_0 < \infty,$$

for all $k > 0$, which, in turn, implies that $x(k), u(k) \to 0$ as $k \to \infty$. □

If A has poles on the unit circle, then the number of control moves H_c necessary to drive the corresponding modes to zero depends on the initial condition (see Chapter 4 for details and an alternative formulation). Not every $H_c \geq N$ may work in this case.

Corollary 7 *Suppose that A has poles on the unit circle. Given any $x(0) \in W_N$,* Controller IHMPCMC *is stabilizing for a sufficiently large H_c.*

Given an initial condition $x(0) \in W$, if the optimization problem (5.9) is feasible, then the infinite output horizon in *Controller IHMPCMC* can be replaced by a finite output horizon with the end constraint $[x_c(k+H_c|k) \quad x_u(k+H_c|k)] = 0$ at each sampling time k. Let

$$\begin{aligned}\Phi_k^F = \textstyle\sum_{i=1}^{H_c-1} x(k+i|k)^T \Gamma_x x(k+i|k) \quad &+ \textstyle\sum_{i=0}^{H_c} \Big[u(k+i|k)^T \Gamma_u u(k+i|k) + \\ &\Delta u(k+i|k)^T \Gamma_{\Delta u} \Delta u(k+i|k)\Big]\end{aligned} \tag{5.10}$$

Theorem 18 *Consider the system represented by (5.2). Assume that H_c is such that* *the optimization problem (5.9) is feasible for a given initial condition $x(0) \in W$. Then the optimization problem (5.9) is equivalent to the following.*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \quad \Phi_k^F + x_s(k+H_c|k)^T P x_s(k+H_c|k) + \epsilon(k)^T Q \epsilon(k)$$

$$\text{subject to} \quad \begin{cases} x_c(k+H_c|k) = 0 & \\ x_u(k+H_c|k) = 0 & \\ u(k+i|k) \in \mathcal{U} & i = 0,\cdots,H_c-1 \\ u(k+i|k) = 0 & i = H_c,\cdots,\infty \\ x(k+i|k) \le \mathcal{X}_{\epsilon(k)} & i = 1,2,\cdots,\infty \\ \epsilon(k) \ge 0 & \end{cases} \tag{5.11}$$

where Φ_k^F is defined by (5.10), P is the solution of the Lyapunov equation $A_s^T P A_s - P = -\Gamma_x^s$, and Γ_x^s is the portion of Γ_x that is associated with x_s.

Proof. Since $u(k+H_c+i|k) = 0, i \ge 0$, J_k is finite if and only if $x_u(k+H_c|k) = 0$ and $x_c(k+H_c|k) = 0$. Thus,

$$\begin{aligned} \sum_{i=H_c}^{\infty} x(k+i|k)^T \Gamma_x x(k+i|k) &= \sum_{i=H_c}^{\infty} x_s(k+i|k)^T \Gamma_x^s x_s(k+i|k) \\ &= x_s(k+H_c|k)^T P x_s(k+H_c|k) \end{aligned}$$

□

5.4 Examples

In this section, we consider two examples. The system in the first example has one pole outside the unit circle and three poles inside the unit circle, two of which are very close to the unit circle. The domain of attractability is determined exactly. A class of controllers (generated by *Controller IHMPCMC*) is constructed to stabilize any initial condition in the domain of attractability. In the second example, the system has two poles outside the unit circle. In addition to approximating the domain of attractability, we give both a subset and a superset, which are very close to each other, of the domain of attractability.

Example 5 Consider a linear model approximating longitudinal dynamics at 3000 ft altitude and 0.6 mach velocity for a modified F-16 aircraft [44].

$$\dot{x} = Ax + Bu$$

where

$$A = \begin{bmatrix} -0.0151 & -60.5651 & 0 & -32.174 \\ -0.0001 & -1.3411 & 0.9929 & 0 \\ -0.00018 & 43.2541 & -0.86939 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2.516 & -13.136 \\ -0.1689 & -0.2514 \\ -17.251 & -1.5766 \\ 0 & 0 \end{bmatrix}$$

The constraints on both inputs are ± 25. The system is discretized with a sampling time of 0.1. Since the system contains only one unstable pole at 1.7252, the domain of attractability for the system is equal to the domain of attractability associated with the unstable pole, i.e. the domain of attractability for the following system:

$$\hat{x}(k+1) = 1.7252\hat{x}(k) + [-91.0626 \quad -15.7785]\hat{u}(k), |\hat{u}(k)|_\infty \leq 1 \ \forall\ k$$

where $\hat{x} = [-0.0002 \;\; 9.5168 \;\; 1.4947 \;\; 0.0013]x$ and $\hat{u} = \frac{u}{25}$. Straightforward calculations yield

$$W_N = \left\{ x : |[-0.0002 \quad 9.5168 \quad 1.4947 \quad 0.0013]x| \leq 106.84 \frac{1 - \left(\frac{1}{1.7252}\right)^N}{1.7252 - 1} \right\}$$

$$W = \{x : |[-0.0002 \quad 9.5168 \quad 1.4947 \quad 0.0013]x| < 147.33\}^8$$

For the initial condition $x(0) = [-65 \;\; 3.5 \;\; 24 \;\; 4.45]^T$, Controller MPC is stabilizing if and only if $H_c \geq 2$. The response for $H_c = 2$ is shown in Figure 5.1. The slow responses are due to the two poles at $0.9992 + 0.0059j$ and $0.9992 - 0.0059j$. Of course, we can speed up the responses by increasing H_c (see Figure 5.2).[9]

Example 6 Consider the following system.

$$A = \begin{bmatrix} -2 & -0.8 \\ -2 & 0.7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Shown in Figure 5.3 are W_{15}, W_{in}, and W_{out} with $T = I$, the identity matrix, and $T = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$. Here W_{in} and W_{out}^u are determined via Equations (5.6) and (5.7) As one can see, T can be chosen to make W_{out} as small as possible and W_{out} with $T = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and W_{in} are very close. Choosing $W = W_{in}$ is a good approximation. For comparison, we also show the domain of attractability for the linear controller which places closed loop poles at 1 and 1.

[8]Notice that W is *open*.

[9]It is interesting to note *in this example* that the 2−norm of the states, i.e. $\sum_{i=0}^{\infty} x(i)^T x(i)$, for $H_c = 2$ is actually *smaller* than that for $H_c = 6$.

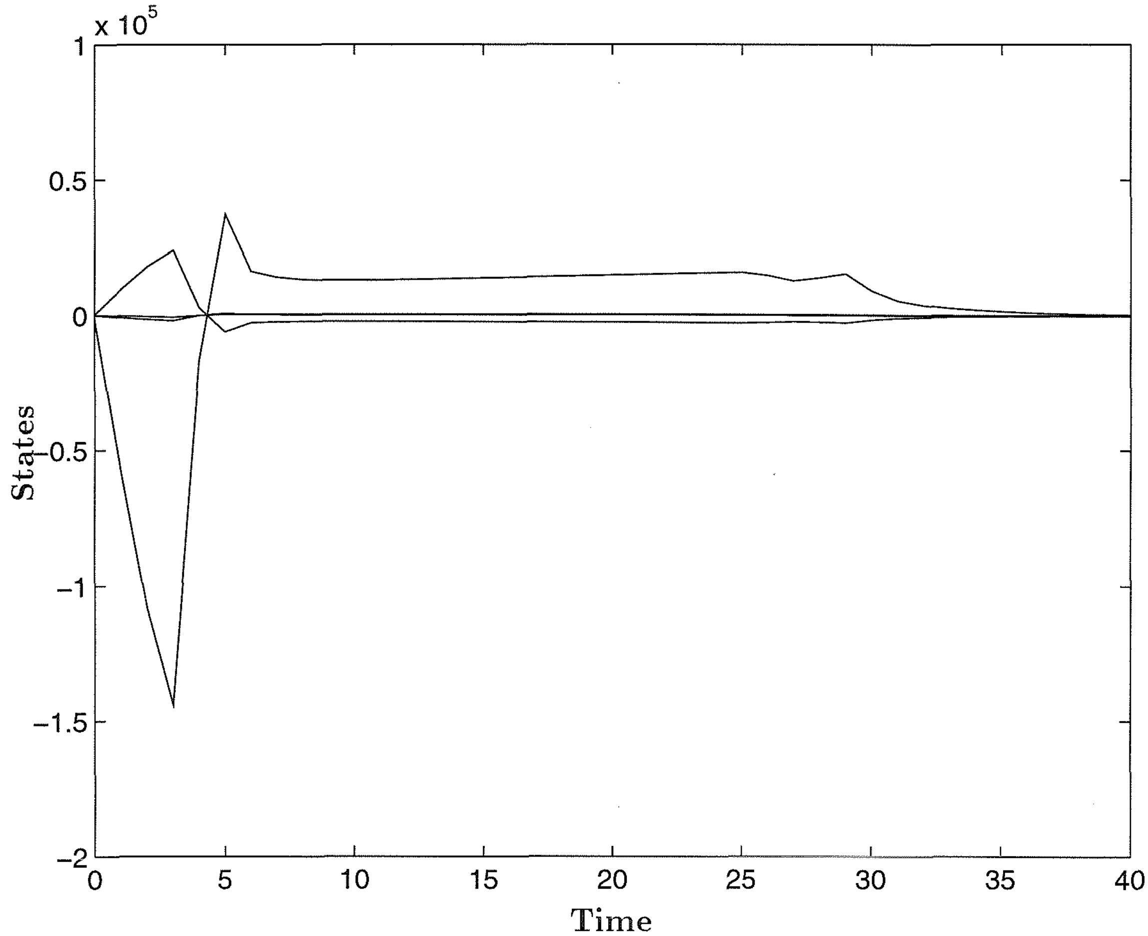

Figure 5.1: Closed loop responses for controller IHMPCMC with $H_c = 2$

5.5 Conclusions

In this chapter we have analyzed the domain of attractability for unstable linear discrete-time systems with hard input constraints and soft constraints. Several methods were presented to characterize the domain of attractability. Although in general the domain of attractability *cannot* be determined exactly, algorithms were introduced to approximate it with an arbitrary accuracy. The major difference of the approach presented here from various approaches existed in the literature is that the domain of attractability does *not* depend on the control law used. We show that, with appropriate choice of the input horizon, the IHMPCMC algorithm stabilizes any initial condition in the domain of attractability.

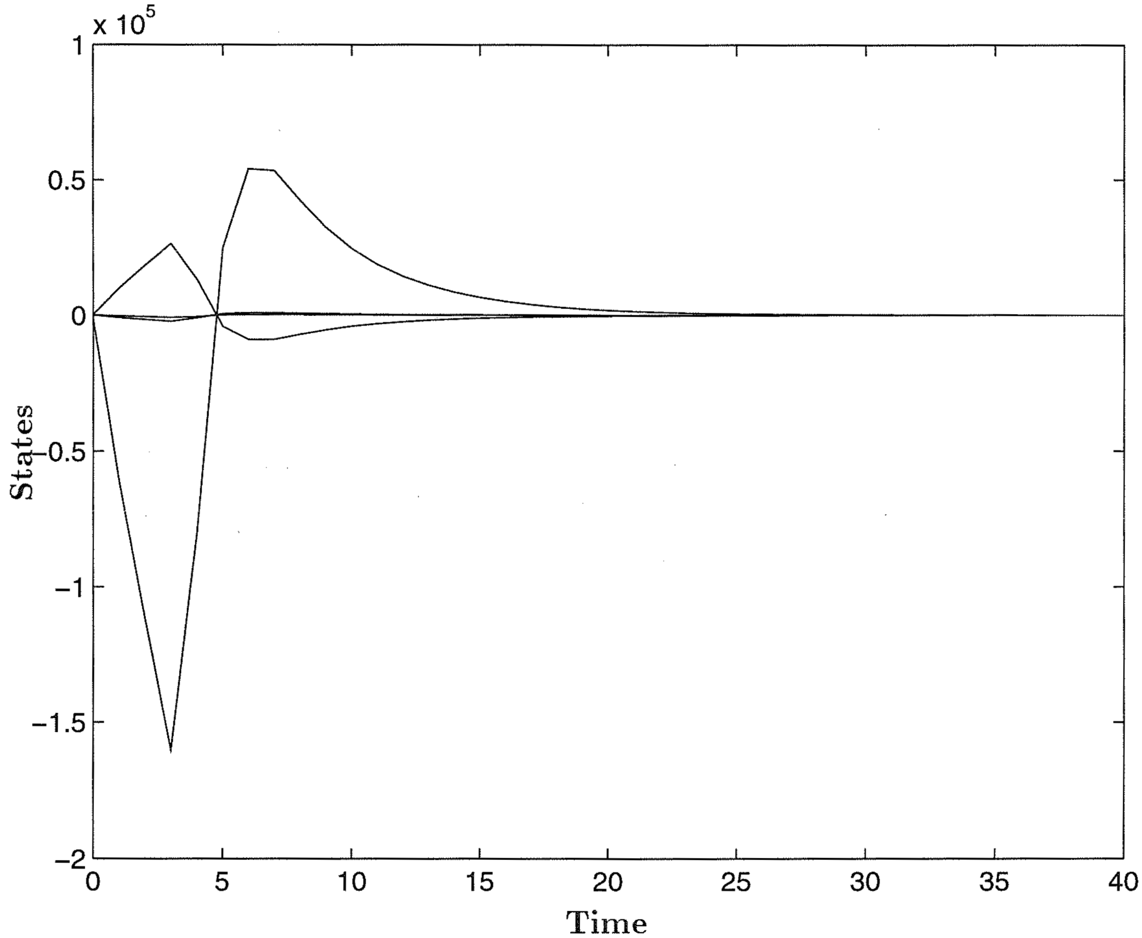

Figure 5.2: Closed loop responses for controller IHMPCMC with $H_c = 6$

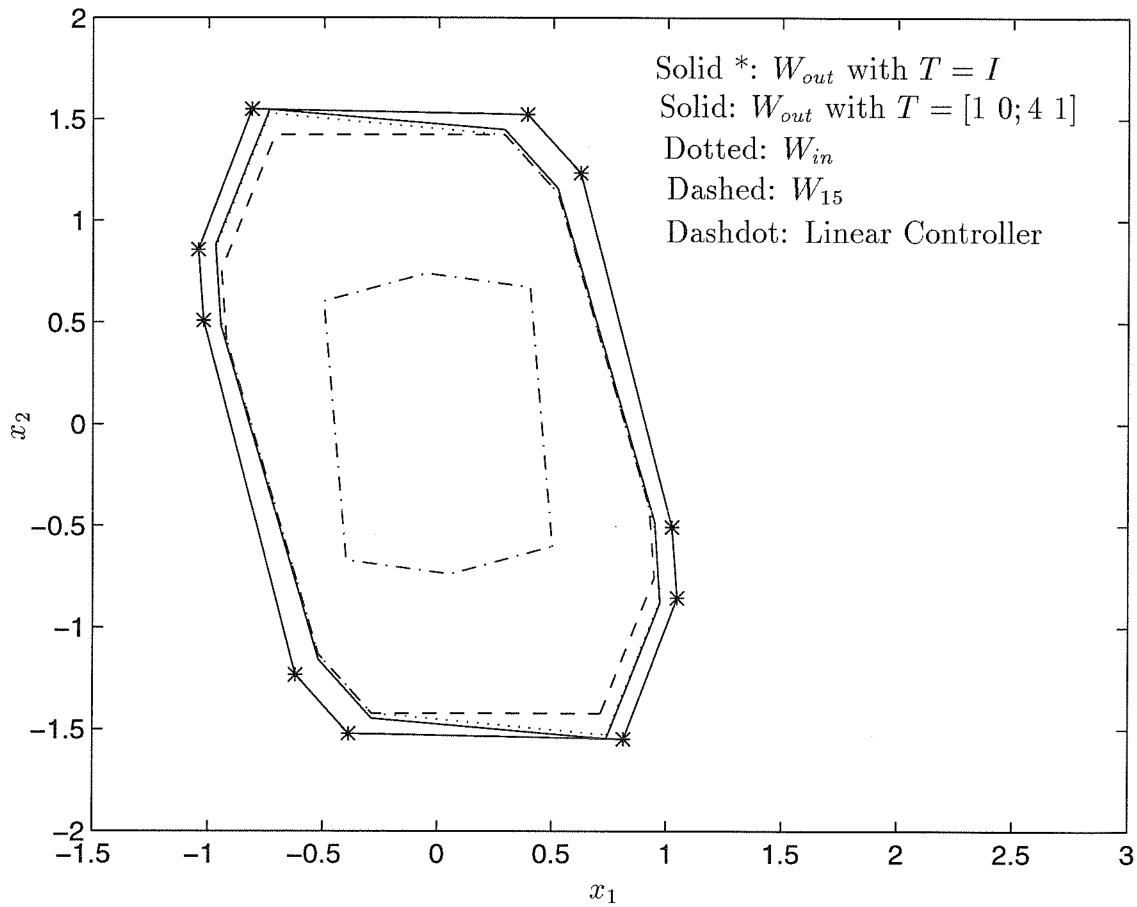

Figure 5.3: Domain of attractability

Chapter 6 Robust Control of Linear Time Varying Systems with Constraints

Summary

In this chapter, we generalize the robust MPC algorithm proposed by Campo and Morari for control of linear uncertain time-varying systems, represented by Finite Impulse Response models, with constraints. We show that with this scheme robust Bounded-Input Bounded-Output stability is guaranteed. Both necessary and sufficient conditions for global asymptotic robust stability are stated. Furthermore, we show that robust global asymptotic stability is preserved for a class of asymptotically constant disturbances entering at the plant output.

Although these results hold for any uncertainty description expressed in the time-domain, there is a trade-off between the generality of the uncertainty description and the computational complexity of the resulting optimization problem. For a broad class of uncertainty descriptions, we show that the optimization problem can be cast as a linear program of moderate size.

6.1 Introduction

All real world control systems must deal with constraints. Although a rich theory has been developed for the robust control of linear systems [73, 21, etc.], *very little* is known about the robust control of linear systems *with constraints.* In this chapter and the next chapter, we use Model Predictive Control (MPC), also known as moving horizon control and receding horizon control, to study this problem. This chapter deals with linear time-varying systems while the next chapter deals with linear time-invariant systems. The basic idea behind MPC and its stability properties in the nominal case were discussed in the previous chapters and will not be repeated here.

Campo and Morari [10, 9] made the first rigorous attempt to extend the MPC concept to the control of uncertain linear time-invariant systems and proposed a robust MPC algorithm. Unfortunately, it is well known (see, for example, [102] for a counter example) that robust stability is not guaranteed with this algorithm. Zafiriou [96] used the contraction mapping principle to derive some necessary and some sufficient conditions for robust stability. However, the conditions are both conservative and difficult to verify. Assuming lower and upper bounds on each impulse response coefficient, Genceli and Nikolaos [32] showed how to determine weights such that robust stability can be guaranteed for a set of Finite Impulse Response (FIR) models. However, often weights may *not* exist even though robust stabilization is *possible* for a set of FIR models. Lee et al. [56] proposed a robust MPC algorithm that minimizes the expectation of a multi-step quadratic objective function for an input-output model with stochastic parameters. Of course, the concept of robust stability cannot be defined in this framework. For a set of linear time-varying systems described in an appropriate way, Kothare [49] proposed a robust MPC algorithm whose optimization problem for the state feedback case can be cast as a set of Linear Matrix Inequalities and showed that global asymptotic stability can be guaranteed if the optimization problem is feasible.

Polak and Yang [75] proposed a receding horizon control strategy for linear continuous time systems with input constraints and proved nominal stability of the closed loop system. Then they showed that robust stability is guaranteed *provided* that the perturbation is sufficiently small. Similar results have been obtained by Mayne and Michalska [63, 64] for nonlinear systems. In all these approaches, the computational issue which is crucial for implementing an MPC algorithm because of its on-line nature was not discussed. Since discussing nonlinear MPC is beyond the scope of this thesis, interested readers are referred to [22] for more reference in the area.

In this chapter, we generalize the robust MPC algorithm introduced by Campo and Morari [10] and demonstrated that this new MPC controller can robustly stabilize *any* set of linear time-varying systems represented by FIR models for which robust stabilization is *possible*. Although the results hold for any uncertainty description expressed in the time-domain, there is a trade-off between the generality of the uncertainty description and the computational complexity of the resulting optimization problem. For a broad class of uncertainty descriptions, we show that the optimization problem can be cast as a linear program of moderate size.

This chapter is organized as follows. In Section 6.2, a robust MPC algorithm is presented and assumptions are stated. In Section 6.3, we show that with this algorithm the closed-loop system is guaranteed to be robustly BIBO stable. Both necessary and sufficient conditions for robust global asymptotic stability are stated. Furthermore, we show that robust global asymptotic stability is preserved for a class of asymptotically constant disturbances entering at the plant output. The extension of the results to integrating systems is also discussed. In Section 6.4, the min-max problem is formulated as a linear program of moderate size for a broad class of uncertainty descriptions. An example is presented in Section 6.5 to demonstrate the characteristics of the proposed method. Section 6.6 concludes the chapter.

Notation Fairly standard notation is used here. x^T denotes the transpose of x. $|\cdot|_1$ denotes the 1−norm on $\Re^n$, $|\cdot|_\infty$ the ∞−norm on $\Re^n$, and $||\cdot||_1$ the 1-norm on $\Re^{n\times n}$, *i.e.* $||x||_1 = \max_j \sum_{i=1}^n |x_{ij}| \; \forall \; x \in \Re^{n\times n}$. It can be easily shown that $||\cdot||_1$ is the operator norm induced by $|\cdot|_1$. I is the identity matrix of dimension of $n\times n$. For $x, y \in \Re^n, x \leq y$ if and only if $x_i \leq y_i, i = 1, 2, \cdots, n$. $\max_{y(k+i|k)} \Leftrightarrow \max_{y(k+i|k)\in Y(k+i|k)}$. $O(\epsilon)$ means in the order of ϵ.

6.2 Preliminary

Consider a stable linear time-varying square system represented by an FIR model

$$y(k+1) = y(k) + \sum_{i=1}^{N} g_i(k+1)\Delta u(k+1-i) + d(k+1) \tag{6.1}$$

where $y(k) \in \Re^n$ and $u(k) \in \Re^n$ are the output and input of the system, respectively, $d(k) \in \Re^n$ is the disturbance, $\Delta u(k) = u(k) - u(k-1)$, and $g(k+1) \triangleq [g_N(k+1) \; \cdots \; g_1(k+1)] \in \Re^{n\times nN}$ is the impulse response coefficient matrix. $g(k+1) \in \Pi$

and Π is a set which is generally obtained from some identification methods. The set of plants generated by Π is given below.

$$\begin{aligned}\mathcal{G} &= \{G : y(k+1) = G(y(k), \Delta u(k-1), \cdots, \Delta u(k-N+1))\} \\ &= \left\{G : y(k+1) = y(k) + \sum_{i=1}^{N} g_i(k+1)\Delta u(k+1-i), g(k+1) \in \Pi\right\}\end{aligned} \tag{6.2}$$

The reason for defining time-varying systems by (6.1) is that a zero steady-state error is possible: the steady-state output value does not change if the input is constant. A linear time-varying system can be defined alternatively as

$$y(k+1) = \sum_{i=1}^{N} g_i(k+1)u(k+1-i) \tag{6.3}$$

The disadvantage is that a zero steady-state offset may not be possible: the steady-state output value varies even if the input is constant.

Define the steady-state gain as $G(k)^{ss} = \sum_{i=1}^{N} g_i(k)$ and

$$G(k)^{ss} \in \Pi^{ss} = \{G(k)^{ss} : G(k)^{ss} = \sum_{i=1}^{N} g_i(k), g(k) \in \Pi\}$$

The control action is generated by *Controller RMPCLTV* which is defined as follows.

Definition 10 Controller RMPCLTV: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$\begin{aligned}J_k &= \min_{\Delta U_k} \max_{y(k+i|k), i=H_s,\cdots,\infty} |\Gamma_y[r(k+i) - y(k+i|k)]|_1 + \sum_{j=0}^{H_c-1} |\Gamma_{\Delta u}\Delta u(k+j|k)|_1 \\ &= \min_{\Delta U_k} \max_{y(k+i|k), i=H_s,\cdots,N+H_c-1} |\Gamma_y[r(k+i) - y(k+i|k)]|_1 + \sum_{j=0}^{H_c-1} |\Gamma_{\Delta u}\Delta u(k+j|k)|_1\end{aligned} \tag{6.4}$$

subject to

$$|\Delta u(k+i|k)| \leq \Delta u^{max}, \qquad i = 0, 1, \cdots, H_c - 1$$

$$u^{min} \leq u(k+i|k) \leq u^{max}, \qquad i = 1, 2, \cdots, H_c$$

where

$\Delta U_k = [\Delta u(k|k)^T \ \cdots \ \Delta u(k+H_c-1|k)^T]^T \in \Re^{nH_c}$;

$\Delta u^{opt}(k+i|k) \in \Re^n$ *denotes the optimal control move at time* $k+i$ *evaluated at time* k *and* $\Delta u(k) \triangleq \Delta u^{opt}(k|k)$;

$y(k+i|k) \in \Re^n$ *is the output at time* $k+i$ *predicted at time* k. $y(k|k) \triangleq y(k)$ *is the measured output at time* k;

$Y(k+i|k) = \{y(k+i|k) : y(k+i|k) = y(k+i-1|k) + g(k+i) \begin{bmatrix} \Delta u(k+i-N) \\ \vdots \\ \Delta u(k+i-1|k) \end{bmatrix},$ $g(k+i) \in \Pi, y(k+i-1|k) \in Y(k+i-1|k)\} \ \forall \ i \geq 1$ *and* $Y(k|k) = \{y(k|k)\}$. *Here we assume implicitly that the disturbance is a step;*

$r(k+i) \in \Re^n$ *is the setpoint at time* $k+i$;

Γ_y *and* $\Gamma_{\Delta u}$ *are positive definite diagonal matrices;*

H_c *is the input horizon; and*

H_s *is the start of the prediction horizon to be minimized. As pointed out by Campo [10] for* $H_s = 1$, *the algorithm does not reject persistent disturbances for systems exhibiting inverse response characteristics. Any control action to reject the disturbance could result in a larger maximum predicted future error than if no control action were taken (as a result of the initial inverse response). Therefore,* H_s *can be adjusted to handle systems with inverse response, dead time, etc. Obviously,* $1 \leq H_s \leq N + H_c - 1$.

Remark 26 *The reason for assuming* $\Gamma_{\Delta u} > 0$ *is as follows: Since no assumption on the set* Π *is imposed, the solution of the optimization problem may not be unique* *if* $\Gamma_{\Delta u} = 0$. *Choosing a sufficiently small positive* $\Gamma_{\Delta u}$ *would ensure the uniqueness of the solution. In most cases, however, we can set* $\Gamma_{\Delta u} = 0$.

Remark 27 *It is generally not possible to have a zero steady-state error for all plants in the set* $\mathcal{G}$. *Using the* $1-$*norm or* $2-$*norm instead of the* $\infty-$*norm temporally may result in an unbounded objective function since the output horizon is infinite.*

Remark 28 *This robust MPC algorithm can also be regarded as a state feedback control strategy. The states are* $y(k)$ *and* $[\Delta u(k-N+1) \ \cdots \ \Delta u(k-1)]$. *They are used to determine the optimal control move* $\Delta u(k)$.

Remark 29 *For SISO systems with* $H_s = 1$, *output constraints,* $y^{min} \leq y(k) \leq y^{max} \ \forall \ k$, *are redundant. This is because the largest deviation from the setpoint is minimized. Another reason for not including the soft output constraints is that the optimal control moves in that case may be zero because* $\infty-$*norm is used temporally.*

Assumptions Throughout the chapter, we make the following assumptions.

Assumption 1 *The real system is stable linear time-varying with* n *inputs and* n *outputs* (i.e. *square*), *and its steady-state gain matrix is nonsingular. Since the system belongs to* $\mathcal{G}$, $\mathcal{G}$ *must contain a model whose steady-state gain matrix is nonsingular.*

Assumption 2 *The setpoint* r *is constant such that a zero steady-state error is feasible for all plants in the set.*[1]

Assumption 3 *The disturbance has the following properties:* $d(k) \rightarrow \bar{d}$ *as* $k \rightarrow \infty$ *and* $\bar{d}$ *is such that a zero steady-state error is feasible for all plants in the set.*

Assumption 4 *The steady-state condition is* $u = 0$ *and* $y = 0$.

6.3 Robust Stability

Let us first prove several lemmas for use later.

[1] If r is time-varying for $k \leq K < \infty$, then we can take the initial time to be K.

Lemma 5 $J_k \leq J_{k-1} - |\Gamma_{\Delta u}\Delta u(k-1)|_1 + |d(k) - d(k-1)|_1$.

Proof. Let

$$\Delta U_k^* = \begin{bmatrix} \Delta u^{opt}(k|k-1) \\ \cdots \\ \Delta u^{opt}(k+H_c-2|k-1) \\ 0 \end{bmatrix}$$

where $\Delta u^{opt}(\bullet|k-1)$ denotes the optimal control moves determined at sampling time $k-1$. Let $Y^*(k+i|k), i = 0, 1, \cdots$, be the set of output values generated by control moves ΔU_k^* for all plants in $\mathcal{G}$. At time k, $y(k)$ is measured and $Y^*(k|k) = \{y(k)\}$. $Y(k|k-1)$ consists all values of output at time k assuming that the disturbance is constant. Since the disturbance may be time-varying, $y(k)$ may not belong to $Y(k|k-1)$. However, $y(k) - \Delta d(k) \in Y(k|k-1)$ where $\Delta d(k) = d(k) - d(k-1)$. Define

$$\begin{aligned} \bar{Y}^*(k+i|k) = \{\bar{y}^*(k+i|k) :\ & \bar{y}^*(k+i|k) = y^*(k+i|k) - \Delta d(k), \\ & y^*(k+i|k) \in Y^*(k+i|k)\}, \quad i = 0, 1, \cdots \end{aligned} \tag{6.5}$$

This together with $\Delta u^*(k+i|k) = \Delta u^{opt}(k+i|k-1), i = 0, 1, \cdots$, yields $\bar{Y}^*(k+i|k) \subseteq Y(k+i|k-1), i = 0, 1, \cdots$. Thus, for all $i \geq 0$, we have

$$\begin{aligned} & \max_{y(k+i|k)\in Y^*(k+i|k)} |\Gamma_y[r - y(k+i|k)]|_1 \\ & \quad \leq \max_{y(k+i|k)\in \bar{Y}^*(k+i|k)} |\Gamma_y[r - y(k+i|k)]|_1 + |\Delta d(k)|_1 \\ & \quad \leq \max_{y(k+i|k)\in Y(k+i|k)} |\Gamma_y[r - y(k+i|k)]|_1 + |\Delta d(k)|_1 \end{aligned}$$

Since ΔU_k^* may not be the optimal solution, we have

$$J_k \leq J_{k-1} - |\Gamma_{\Delta u}\Delta u(k-1)|_1 + |d(k) - d(k-1)|_1$$

□

Remark 30 *In Campo's formulation [10], the plant is assumed to be time-invariant. Lemma 5 does not hold since the worst-case plant changes from sampling time to sampling time.*

Lemma 6 *Suppose* $\Delta u(k-i) \sim O(\epsilon), i = 1, 2, \cdots, N$, *and* $|d(k) - d(k-1)|_1 \sim O(\epsilon)$ *where* ϵ *is an arbitrarily small positive constant. For* $H_s = N$ *and* $H_c = 1$, *there exists a constant* $\beta > 0$ *such that*

$$J_k \geq J_{k-1} - \beta |\Delta u(k)|_1 + O(\epsilon) \tag{6.6}$$

for a sufficiently small $\Gamma_{\Delta u}$.

Proof. From definition of $Y(k+i|k), i = 1, 2, \cdots$, we obtain $Y(k+N-1|k) = \{y(k+N-1|k) : y(k+N-1|k) = y(k) + G(k)^{ss}\Delta u(k|k) + O(\epsilon), G(k)^{ss} \in \Pi^{ss}\}$ where the term $O(\epsilon)$ denotes the effect from $\Delta u(k-i), i = 1, 2, \cdots, N$. This gives

$$\begin{aligned} J_{k-1} &= \max_{y(k+N-2|k-1)} [|\Gamma_y[r - y(k+N-2|k-1)]|_1 + |\Gamma_{\Delta u}\Delta u(k-1)|_1] \\ &= |\Gamma_y(r - y(k))|_1 + O(\epsilon) \end{aligned}$$

and

$$\begin{aligned} J_k &= \min_{\Delta u(k|k)} \max_{y(k+N-1|k)} [|\Gamma_y[r - y(k+N-1|k)]|_1 + |\Gamma_{\Delta u}\Delta u(k|k)|_1] + O(\epsilon) \\ &= \min_{\Delta u(k|k)} \max_{G(k)^{ss}\in\Pi^{ss}} [|\Gamma_y[r - y(k) - G(k)^{ss}\Delta u(k|k)]|_1 + |\Gamma_{\Delta u}\Delta u(k|k)|_1] + O(\epsilon) \\ &\geq \min_{\Delta u(k|k)} [|\Gamma_y[r - y(k) - G_0^{ss}\Delta u(k|k)]|_1 + |\Gamma_{\Delta u}\Delta u(k|k)|_1] + O(\epsilon) \\ &\qquad\qquad \text{(for some nonsingular } G_0^{ss} \in \Pi^{ss}) \\ &= |\Gamma_y[r - y(k) - G_0^{ss}\Delta u(k)]|_1 + |\Gamma_{\Delta u}\Delta u(k)|_1 + O(\epsilon) \\ &\geq |\Gamma_y[r - y(k)]|_1 - |\Gamma_y G_0^{ss}\Delta u(k)|_1 + |\Gamma_{\Delta u}\Delta u(k)|_1 + O(\epsilon) \\ &= J_{k-1} - |\Gamma_y G_0^{ss}\Delta u(k)|_1 + |\Gamma_{\Delta u}\Delta u(k)|_1 + O(\epsilon) \\ &\geq J_{k-1} - (||\Gamma_y G_0^{ss}||_1 - \underline{\gamma}_u)|\Delta u(k)|_1 + O(\epsilon) \\ &= J_{k-1} - \beta|\Delta u(k)|_1 + O(\epsilon) \end{aligned}$$

where γ_u is the smallest diagonal element of $\Gamma_{\Delta u}$. Since G_0^{ss} is nonsingular and $\Gamma_y > 0$, choosing $\Gamma_{\Delta u}$ sufficiently small guarantees $\beta > 0$. □

Lemma 7 *Suppose $\Delta u(k-i) \sim O(\epsilon), i = 1, 2, \cdots, N$, and $|d(k) - d(k-1)|_1 \sim O(\epsilon)$ where ϵ is an arbitrarily small positive constant. For $H_s = N$ and $H_c = 1$, there exist positive constants $\gamma_1 < 1$ and γ_2 such that*

$$J_k \leq \max(\gamma_1 J_{k-1}, J_{k-1} - \gamma_2) + O(\epsilon) \tag{6.7}$$

for a sufficiently small $\Gamma_{\Delta u}$ if there exists some nonsingular $G_0^{ss} \in \Pi^{ss}$ such that

$$\max_{G(k)^{ss} \in \Pi^{ss}} ||I - \Gamma_y G(k)^{ss} (G_0^{ss})^{-1} \Gamma_y^{-1}||_1 = \lambda < 1 \tag{6.8}$$

Proof. Follow the similar arguments as in the proof of Lemma 6, we have

$$J_{k-1} = |\Gamma_y (r - y(k))|_1 + O(\epsilon)$$

and

$$J_k = \min_{\Delta u(k|k)} \max_{G(k)^{ss} \in \Pi^{ss}} [|\Gamma_y [r - y(k) - G(k)^{ss} \Delta u(k|k)]|_1 + |\Gamma_{\Delta u} \Delta u(k|k)|_1] + O(\epsilon)$$

Let $\Delta u^*(k|k) = \alpha (G_0^{ss})^{-1} [r - y(k)], 0 < \alpha \leq 1$. α can be chosen such that $\Delta u^*(k|k)$ is feasible: Choosing α sufficiently small guarantees $|\Delta u^*(k|k)| \leq \Delta u^{max}$. Since $G_0^{ss} \in \Pi^{ss}$, $u(k-1) + (G_0^{ss})^{-1}[r - y(k)]$ is the steady-state input for some plant in the set. By assumption that the steady-state input for all plants in the set does not violate the constraints, we have $u^{min} \leq u(k-1) + (G_0^{ss})^{-1}[r - y(k)] \leq u^{max}$. This together with $u^{min} \leq u(k-1) \leq u^{max}$ gives $u^{min} \leq u(k-1) + \alpha (G_0^{ss})^{-1}[r - y(k)] \leq u^{max}$ for all $0 < \alpha \leq 1$. $\Delta u(k|k) = \Delta u^*(k|k)$ may not be the optimal solution. We have

$$J_k \leq \max_{G(k)^{ss} \in \Pi^{ss}} |\Gamma_y [r - y(k) - G(k)^{ss} \Delta u^*(k|k)]|_1 + |\Gamma_{\Delta u} \Delta u^*(k|k)|_1 + O(\epsilon)$$

$$
\begin{aligned}
&= \max_{G(k)^{ss}\in\Pi^{ss}} |\Gamma_y[r - y(k) - \alpha G(k)^{ss}(G_0^{ss})^{-1}(r - y(k))]|_1 \\
&\qquad +|\Gamma_{\Delta u}\alpha(G_0^{ss})^{-1}(r - y(k))|_1 + O(\epsilon) \\
&= \max_{G(k)^{ss}\in\Pi^{ss}} |(I - \alpha\Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1})\Gamma_y(r - y(k))|_1 \\
&\qquad +\alpha|\Gamma_{\Delta u}(G_0^{ss})^{-1}\Gamma_y^{-1}\Gamma_y(r - y(k))|_1 + O(\epsilon) \\
&\leq \max_{G(k)^{ss}\in\Pi^{ss}} ||I - \alpha\Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1|\Gamma_y(r - y(k))|_1 \\
&\qquad +\alpha||\Gamma_{\Delta u}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1|\Gamma_y(r - y(k))|_1 + O(\epsilon) \\
&\leq \gamma|\Gamma_y(r - y(k))|_1 + O(\epsilon)
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma &= \max_{G(k)^{ss}\in\Pi^{ss}} ||I - \alpha\Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 + \alpha||\Gamma_{\Delta u}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 \\
&\leq \max_{G(k)^{ss}\in\Pi^{ss}} ||\alpha I - \alpha\Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 + (1-\alpha)||I||_1 + \alpha||\Gamma_{\Delta u}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 \\
&\qquad\qquad (\text{ since } 0 < \alpha \leq 1) \\
&\leq \alpha \max_{G(k)^{ss}\in\Pi^{ss}} ||I - \Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 + (1-\alpha) + \alpha||\Gamma_{\Delta u}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 \\
&= 1 - (1 - \lambda - \mu)\alpha
\end{aligned}
$$

where $\mu = ||\Gamma_{\Delta u}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1$. Since $\lambda < 1$, choosing $\Gamma_{\Delta u}$ sufficiently small guarantees $\gamma < 1$. Thus we have

$$J_k \leq \gamma J_{k-1} + O(\epsilon)$$

Notice that γ is *not* a constant and can be arbitrarily close to 1. To obtain the constants γ_1 and γ_2 as in the statement of the lemma, let us consider the following two cases.

Case 1—$\alpha = 1$. $\alpha = 1$ does not result in any constraint violation on Δu. Then letting $\alpha = 1$ gives

$$J_k \leq \gamma_1 J_{k-1} + O(\epsilon)$$

where $\gamma_1 = \lambda + \mu < 1$ is constant.

Case 2—$0 < \alpha < 1$. $\alpha = 1$ results in constraint violations on Δu. Then choosing

$$\alpha = \frac{\min(\Delta u^{max})}{|(G_0^{ss})^{-1}(r - y(k))|_\infty}$$

where $\min(\Delta u^{max}) > 0$ denotes the smallest element of Δu^{max}, does not result in any constraint violation and $\alpha < 1$. Using the fact $J_{k-1} = |\Gamma_y(r - y(k))|_1 + O(\epsilon)$ and G_0^{ss} is nonsingular, after a few lines of algebra, we get

$$\alpha \geq \frac{\nu}{J_{k-1} + O(\epsilon)}$$

where $\nu > 0$ is a constant. The following completes the proof.

$$\begin{aligned}
J_k &\leq \gamma J_{k-1} + O(\epsilon) \\
&= (1 - (1 - \gamma_1)\alpha)J_{k-1} + O(\epsilon) \\
&= J_{k-1} - (1 - \gamma_1)\alpha J_{k-1} + O(\epsilon) \\
&\leq J_{k-1} - (1 - \gamma_1)\frac{\nu}{J_{k-1} + O(\epsilon)} J_{k-1} + O(\epsilon) \\
&= J_{k-1} - (1 - \gamma_1)\nu + (1 - \gamma_1)\nu \frac{O(\epsilon)}{J_{k-1} + O(\epsilon)} + O(\epsilon) \\
&= J_{k-1} - \gamma_2 + O(\epsilon)
\end{aligned}$$

where $\gamma_2 = (1 - \gamma_1)\nu > 0$ is a constant. □

The following theorem states that robust BIBO stability is guaranteed for *all* values of tuning parameters.

Theorem 19 *Assume that there are no input constraints and that there are no disturbances. Then the closed-loop system is guaranteed to be robustly BIBO stable for all values of the tuning parameters $H_c, H_s, \Gamma_{\Delta u}$ and Γ_y.*

Proof. By Lemma 5, we have $J_{k+1} \leq J_k - |\Gamma_{\Delta u}\Delta u(k)|_1$. Thus the optimal value of the objective function is bounded for all times. Since $\Gamma_y > 0$, the output must be bounded for all times. Also, $J_{k+1} \leq J_0 - \sum_{i=0}^{k} |\Gamma_{\Delta u}\Delta u(i)|_1 \leq J_0 - |\Gamma_{\Delta u}u(k)|_1 \Rightarrow |\Gamma_{\Delta u}u(k)|_1 \leq J_0 < \infty$. Since $\Gamma_{\Delta \dot{u}} > 0$, $u(k)$ must also be bounded for all values of k.

Therefore, the closed-loop system is robustly BIBO stable. □

Since the plant is stable, robust BIBO stability is trivially satisfied if the input is constrained. Theorem 19 is only meaningful if the input constraints are not present. Theorem 19 demonstrates the power of the proposed robust MPC algorithm. However, generally we would like that the output approaches the setpoint asymptotically, *i.e.* robust global asymptotic stability is preferred. The following theorem establishes a sufficient condition for robust global asymptotic stability.

Theorem 20 *The closed-loop system is robustly globally asymptotically stable for* $H_s = N$*, all values of* H_c*, and a sufficiently small* $\Gamma_{\Delta u}$ *if there exists nonsingular* $G_0^{ss} \in \Pi^{ss}$ *such that*

$$\max_{G(k)^{ss} \in \Pi^{ss}} ||I - \Gamma_y G(k)^{ss} (G_0^{ss})^{-1} \Gamma_y^{-1}||_1 < 1 \tag{6.9}$$

Proof. We only need to show that the theorem holds for $H_c = 1$: the theorem clearly holds for $H_c > 1$ if it holds for $H_c = 1$. The optimal value of the objective function becomes

$$J_k = \min_{\Delta u(k|k)} \max_{y(k+N-1|k)} [|\Gamma_y[r - y(k+N-1|k)]|_1 + |\Gamma_{\Delta u} \Delta u(k|k)|_1]$$

Suppose J_k does not approach zero as $k \to \infty$. By Lemma 5, we have

$$J_k \leq J_{k-1} - |\Gamma_{\Delta u} \Delta u(k-1)|_1 + |d(k) - d(k-1)|_1$$

By assumption, $|d(k) - d(k-1)|_1 \to 0$ as $k \to \infty$. Therefore, $\Delta u(k) \to 0$ as $k \to 0$; otherwise, the second term would eventually catch up with the first term and J_k would approach zero asymptotically. By Lemma 6, we have

$$|\Delta u(k)|_1 \geq \frac{1}{\beta}(J_{k-1} - J_k) + O(\epsilon(k))$$

This together with Lemma 7 gives

$$|\Delta u(k)|_1 \geq \frac{1}{\beta}\min(\gamma_2, (1-\gamma_1)J_{k-1}) + O(\epsilon(k))$$

Since $\epsilon(k) \to 0$ as $k \to \infty, \gamma_1 < 1, \gamma_2 > 0, \beta > 0$ and $J_{k-1} \neq 0$, $\Delta u(k)$ does not approach zero as $k \to \infty$ and we have a contradiction. Therefore, $J_k \to 0$ as $k \to \infty$ and the tracking error approaches zero asymptotically. □

Remark 31 *For SISO systems, the sufficient condition (6.9) becomes that all plants in the set have the same steady-state gain sign. Thus it is also necessary for robust global asymptotic stability. For MIMO systems, the condition is trivially satisfied if there is no uncertainty associated with the nominal model and the steady-state gain matrix of the nominal model is not singular.*

Theorem 20 is shown only for $H_c = N$. In general, H_c should be chosen as small as possible to improve performance. The following corollary states that smaller values of H_c can be chosen to insure robust global asymptotic stability.

Corollary 8 *Let* $\Pi^{ss*} = \{G(k)^{ss} : G(k)^{ss} = \sum_{i=1}^{j} g_i(k), N^* \leq j \leq N, g(k) \in \Pi\}$. *Then the closed-loop system is robustly globally asymptotically stable for all* $H_s \geq N^*$, *all values of* H_c, *and a sufficiently small* $\Gamma_{\Delta u}$ *if there exists nonsingular* $G_0^{ss} \in \Pi^{ss*}$ *such that*

$$\max_{G(k)^{ss} \in \Pi^{ss*}} ||I - \Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1}||_1 < 1 \tag{6.10}$$

A necessary condition for robust global asymptotic stability is stated in the following theorem.

Theorem 21 *The closed-loop system is not robustly globally asymptotically stable for any values of* H_s, H_c *and* $\Gamma_{\Delta u}$ *if there does not exist* $G_0^{ss} \in \Re^{n\times n}$ *such that*

$$\max_{G(k)^{ss} \in \Pi^{ss}} ||(I - \Gamma_y G(k)^{ss}(G_0^{ss})^{-1}\Gamma_y^{-1})r||_1 < |r|_1 \text{ for some } r \tag{6.11}$$

Proof. WLOG, assume that the initial condition is zero and that there are no disturbances. From the definition of $Y(k+i|k)$, we have

$$\left\{y(N+H_c|1) : y(N+H_c|1) = G(k)^{ss}\sum_{i=1}^{H_c}\Delta u(i|1), G(k)^{ss}\in\Pi^{ss}\right\} \subseteq Y(N+H_c|1)$$

Then

$$\begin{aligned} J_1 &\geq \min_{\Delta U_1} \max_{y(N+H_c|1)} |\Gamma_y[r-y(N+H_c|1)]|_1 + \sum_{j=0}^{H_c-1}|\Gamma_{\Delta u}\Delta u(1+j|1)|_1 \\ &= \min_{\Delta U_1} \max_{G(k)^{ss}\in\Pi^{ss}} \left|\Gamma_y[r - G(k)^{ss}\sum_{i=1}^{H_c}\Delta u(i|1)]\right|_1 + \sum_{j=0}^{H_c-1}|\Gamma_{\Delta u}\Delta u(1+j|1)|_1 \\ &= \min_{\Delta U_1} \max_{G(k)^{ss}\in\Pi^{ss}} \left|\Gamma_y r - \Gamma_y G(k)^{ss}\sum_{i=1}^{H_c}\Delta u(i|1)\right|_1 + \sum_{j=0}^{H_c-1}|\Gamma_{\Delta u}\Delta u(1+j|1)|_1 \end{aligned}$$

Assume that the optimal control move is such that $\sum_{i=0}^{H_c-1}\Delta u^{opt}(1+i|1) = (G_0^{ss})^{-1}r$ for some $G_0^{ss}\in\Re^{n\times n}$. By condition (6.11), there exists r such that $J_1 \geq |\Gamma_y r|_1$. Since $J_1 = |\Gamma_y r|_1$ if $\Delta U_1 = 0$ and $J_1 > |\Gamma_y r|_1$ if $\Delta U_1 \neq 0$, $\Delta U_1 = 0$ is the optimal solution. Therefore, no control actions are taken for some setpoint change and robust global asymptotic stability is not guaranteed. □

A necessary condition which is easier to check is stated in the following theorem.

Theorem 22 *The closed-loop system is not globally asymptotically robustly stable for any values of $H_c, H_s, \Gamma_{\Delta u}$ and Γ_y if $G(k)^{ss}$ is singular for some $G(k)^{ss}\in\Pi^{ss}$.*

Proof. From the proof of Theorem 21, we have

$$J_1 \geq \min_{\Delta U_1}\max_{G(k)^{ss}\in\Pi^{ss}}\left[|\Gamma_y r - \Gamma_y G(k)^{ss}\sum_{i=0}^{H_c-1}\Delta u(1+i|1)|_1 + \sum_{j=0}^{H_c-1}|\Gamma_{\Delta u}\Delta u(1+j|1)|_1\right]$$

If $G(k)^{ss}$ is singular for some $G(k)^{ss}\in\Pi^{ss}$, then $J_1 \geq |\Gamma_y r|_1 + \sum_{j=0}^{H_c-1}|\Gamma_{\Delta u}\Delta u^{opt}(1+j|1)|_1$ for some $r\in\Re^n$. Since $J_1 = |\Gamma_y r|_1$ if $\Delta U_1 = 0$ and $J_1 > |\Gamma_y r|_1$ if $\Delta U_1 \neq 0$, $\Delta U_1 = 0$ is the optimal solution. Thus the output does not approach the setpoint asymptotically. □

Integrating Systems The SISO integrating systems can be treated in the same manner by replacing Δu in Equation (1) by u. All results presented follow immediately with one exception: For stable systems, the class of asymptotically constant disturbances entering at the plant *output* is the same as the class of asymptotically constant disturbances entering at the plant *input*. This is clearly not the case for integrating systems. Although all results hold for stable systems when the disturbances enter at the plant input instead of the plant output, whether this is the case for integrating systems needs to be investigated. One difficulty may arise in extending the results to MIMO integrating systems. There may be an integrator between one input and output 1 while there is no integrator between this input and output 2. A different system description may have to be used.

6.4 Computation of Control Moves

The results proven in the previous section hold for *any* uncertainty description expressed in the time-domain. However, there is a trade-off between the generality of the uncertainty description and the computational complexity of the resulting min-max problem. The more general the uncertainty description is, the more expensive is the computation. Here we consider an uncertainty description for which a good compromise between the generality of uncertainty descriptions and the computational complexity is reached. Because of the space limitation, some details are omitted.

The set Π is given by

$$\Pi = \left\{ g : g = \bar{g} + \sum_{i=1}^{l} \Delta_i V_i, \bar{g} \in \Re^{n \times nN}, V_i \in \Re^{n \times nN}, \Delta_i \in \mathbf{\Delta},\ i = 1, \cdots, l \right\} \tag{6.12}$$

where

$$\mathbf{\Delta} = \{\Delta : \Delta = \operatorname{diag}\{\delta_1, \cdots, \delta_n\} \text{ and } |\delta_i| \leq 1, i = 1, \cdots, n\}$$

We want to show that the min-max problem can be cast as a linear program of moderate size. The following lemma can be shown easily.

Lemma 8 $\max_{\Delta_i \in \mathbf{\Delta}} |x + \Delta_i y|_1 = |x|_1 + |y|_1 \ \forall\ x, y \in \Re^n$.

An important step in casting the optimization problem (6.4) (which is a min-max problem) as a linear program is that the special structure of the uncertainty description (6.12) allows us to remove the "max" operation. WLOG, assume $\Gamma_y = I$.[2] Let

$$\Delta v_{k+i} = \left[\Delta u(k+i-N)^T \ \cdots \ \Delta u(k|k)^T \ \cdots \Delta u(k+i-1|k)^T\right]^T \in \Re^{nN}$$

We have

$$\max_{y(k+i|k)} |r - y(k+i|k)|_1 = \max_{y(k+i-1|k)} \ \max_{g(k+i)\in\Pi} |r - y(k+i-1|k) - g(k+i)\Delta v_{k+i}|_1$$

$$\begin{aligned}
&= \max_{y(k+i-1|k)} \ \max_{\Delta_p(k+i)\in\mathbf{\Delta}} \left| r - y(k+i-1|k) - \left(\bar{g} + \sum_{p=1}^{l} \Delta_p(k+i)V_p\right)\Delta v_{k+i}\right|_1 \\
&= \max_{y(k+i-1|k)} \ \max_{\Delta_p(k+i)\in\mathbf{\Delta}} \left| r - y(k+i-1|k) - \bar{g}\Delta v_{k+i} - \sum_{p=1}^{l} \Delta_p(k+i)V_p\Delta v_{k+i}\right|_1 \\
&= \max_{y(k+i-1|k)} |r - y(k+i-1|k) - \bar{g}\Delta v_{k+i}|_1 + \sum_{p=1}^{l} |V_p \Delta v_{k+i}|_1
\end{aligned}$$

The first three equalities follow from the definition while the last equality follows from Lemma 8. Repeating this process i times gives

$$\max_{y(k+i|k)} |r - y(k+i|k)|_1 = \left| r - y(k) - \bar{g}\sum_{j=1}^{i} \Delta v_{k+j}\right|_1 + \sum_{j=1}^{i}\sum_{p=1}^{l} |V_p \Delta v_{k+j}|_1 \qquad (6.13)$$

Thus the optimization problem (6.4) is equivalent to

$$\min_{\Delta U_k} \theta \qquad (6.14)$$

[2]Since diagonal Γ_y commutes with Δ, i.e. $\Gamma_y \Delta = \Delta \Gamma_y, \Delta \in \mathbf{\Delta}$.

subject to

$$\left|r - y(k) - \bar{g}\sum_{j=1}^{i} \Delta v_{k+j}\right|_1 + \sum_{j=1}^{i}\sum_{p=1}^{l} |V_p \Delta v_{k+j}|_1 + \sum_{j=0}^{H_c-1} |\Gamma_u \Delta u(k+j|k)|_1 \leq \theta$$

$$i = H_s, \cdots, N + H_c - 1$$

$$|\Delta u(k+i|k)| \leq \Delta u^{max} \qquad i = 0, \cdots, H_c - 1$$

$$u^{min} \leq u(k+i|k) \leq u^{max} \qquad i = 0, \cdots, H_c - 1$$

Define

$$\begin{aligned}
\alpha_{ij} &= \left|r_j - y_j(k) - \bar{g}(j,:)\sum_{p=1}^{i} \Delta v_{k+p}\right| \\
\beta_{iop} &= |V_o(p,:)\Delta v_{k+i}| \\
\Delta z_{ij} &= |\Delta u_i(k-1+j|k)|_1
\end{aligned}$$

where $X(i,:)$ denotes the i^{th} row of X. We have

$$\min_{\Delta U_k} \theta \tag{6.15}$$

subject to

$$\sum_{j=1}^{n} \alpha_{ij} + \sum_{j=1}^{i}\sum_{o=1}^{l}\sum_{p=1}^{n} \beta_{jop} + \sum_{i=1}^{n}\sum_{j=1}^{H_c} \Delta z_{ij} \leq \theta \qquad i = H_s, \cdots, N + H_c - 1$$

$$-\alpha_{ij} \leq r_j - y_j(k) - \bar{g}\sum_{p=1}^{i} \Delta v_{k+p} \leq \alpha_{ij} \qquad i = H_s, \cdots, N + H_c - 1; \quad j = 1, \cdots, n$$

$$-\beta_{iop} \leq V_o(p,:)\Delta v_{k+i} \leq \beta_{iop}$$

$$i = 1, \cdots, N + H_c - 1; \quad o = 1, \cdots, l; \quad p = 1, \cdots, n$$

$$-\Delta z_{ij} \leq \Delta u_i(k-1+j|k) \leq \Delta z_{ij} \qquad i = 1, \cdots, n; \quad j = 1, \cdots, H_c$$

$$\Delta z_{ij} \leq \Delta u_i^{max} \qquad i = 1, \cdots, n; \quad j = 1, \cdots, H_c$$

$$u_j^{min} \leq u_j(k+i|k) \leq u_j^{max} \qquad i = 0, \cdots, H_c - 1; \quad j = 1, \cdots, n$$

Notice that we have replaced the equality constraints for α and β by the inequality constraints. The reason that we can do this is that these inequality constraints must occur as equality constraints at the optimal solution. For example, suppose that the inequality constraint for $\alpha_{H_s n}$ is not an equality, then we can reduce the value of θ by reducing the value of $\alpha_{H_s n}$ without violating any constraints. Thus we get a contradiction. Also the optimal solution must have $\Delta z_{ij} = |\Delta u_i(k-1+j|k)|$. Otherwise, the value of θ can be reduced by reducing the value of Δz_{ij} without violating any constraints. The above optimization problem can be written as the standard linear program

$$\min_x fx \qquad \text{subject to} \quad Ax \leq b \tag{6.16}$$

The number of constraints is at most $(2n+1)(N+H_c-H_s)+2nd(N+H_c-1)+5H_cn$ and the number of variables is at most $1+n(N+H_c-H_s)+nd(N+H_c-1)+2H_cn$. Both these numbers are *linear* in parameters and the size of the linear program is moderate.

6.5 Example

In this section, we present an example to demonstrate the characteristics of the proposed method. The main point of this example is that robust global asymptotic stability is *guaranteed.*

Example 7 The set of models is described as follows

$$\mathcal{G} = \{G(q) : G(q) = \delta_2 G_0(q) + \delta_1 \left[G_0(q) - G_1(q)\right], 0 \leq \delta_1 \leq 0.5 \text{ and } 0.5 \leq \delta_2 \leq 1.5\} \tag{6.17}$$

where $G_0(q) = \frac{0.75}{q-0.75}$ and $G_1 = \frac{0.75(-q+1.8)}{(q-0.75)(q-0.2)}$. Here δ_1 and δ_2 can be interpreted as follows: δ_1 accounts for possible unmodelled dynamics while δ_2 accounts for the gain uncertainty.

$\mathcal{G}$ can be put into the following form:

$$\mathcal{G} = \left\{ G : G = \left[\frac{5}{4} G_0^{ss}(q) - \frac{1}{4} G_1(q) \right] + \bar{\delta}_2 \frac{1}{2} G_0^{ss}(q) + \bar{\delta}_1 \frac{1}{4} \left[G_0^{ss}(q) - G_1(q) \right], |\bar{\delta}_i| \leq 1 \right\}$$

G_0 and G_1 are approximated by FIR models of order 15 ($N = 15$) and can then be represented by (6.1). We can put the impulse response coefficient set Π into the form (6.12) with $d = 2$ and $\mathbf{\Delta} = \{\Delta : \Delta \in \Re \text{ and } |\Delta| \leq 1\}$. Thus the optimization problem (6.4) can be cast as a linear program. Since the steady-state gain for all plants in the set is positive, by Theorem 20, robust asymptotic stability is guaranteed for $H_s = N$. Furthermore, by Corollary 1, robust asymptotic stability is guaranteed for all $H_s \geq 3$. The values of tuning parameters are $H_s = 3, H_c = 2$, and $\Gamma_{\Delta u} = 0$. The resulting linear program has 106 constraints and 46 variables. The input is constrained between the saturation limits ± 1.

Figure 6.1 shows the output response for a unit-step setpoint change for a linear time-varying system whose parameter variations are shown in Figure 6.2. Since the class of LTI systems can be considered as a subclass of LTV systems, we can apply the robust algorithm presented in this chapter to LTI systems as well and it follows that stability can be guaranteed. However, the performance may be poor (see Chapter 7 for more details). Figure 6.3 shows the output responses for a unit-step setpoint change for several values of δ_1 and δ_2. An additive disturbance resulting from a unit-step disturbance going through a lag of $\frac{0.4}{q-0.6}$ is introduced at the output. As we can see from Figure 6.4, the disturbance is rejected asymptotically.

6.6 Conclusions

In this chapter, we have generalized the robust MPC algorithm proposed by Campo and Morari [10] for control of uncertain linear time-varying systems (represented by FIR models) with constraints. We showed that with this scheme robust Bounded-Input Bounded-Output stability is guaranteed for all values of tuning parameters. Both necessary and sufficient conditions for global asymptotic robust stability were

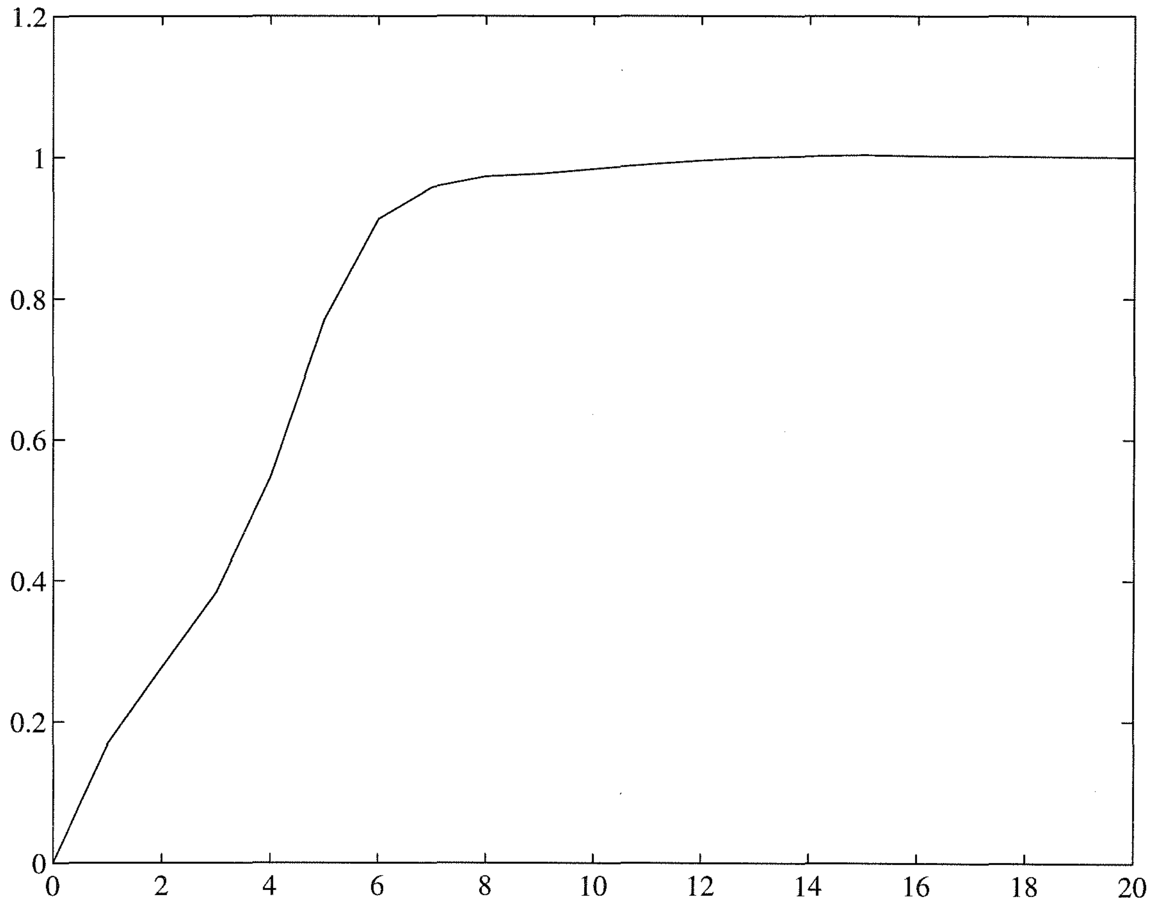

Figure 6.1: Responses for a set-point change

stated. Furthermore, we showed that robust global asymptotic stability is preserved for a class of asymptotically constant disturbances entering at the plant output.

Although these results hold for any uncertainty description expressed in the time-domain, there is a trade-off between the generality of the uncertainty description and the computational complexity of the resulting optimization problem. For a broad class of uncertainty descriptions, we show that the optimization problem can be cast as a linear program of moderate size. We also discussed the extension of these results to integrator systems.

In principle, we can apply the robust MPC algorithm presented in this chapter to control of uncertain linear time-invariant systems and show that stability can be guaranteed under the same conditions. However, this often produces conservative results.

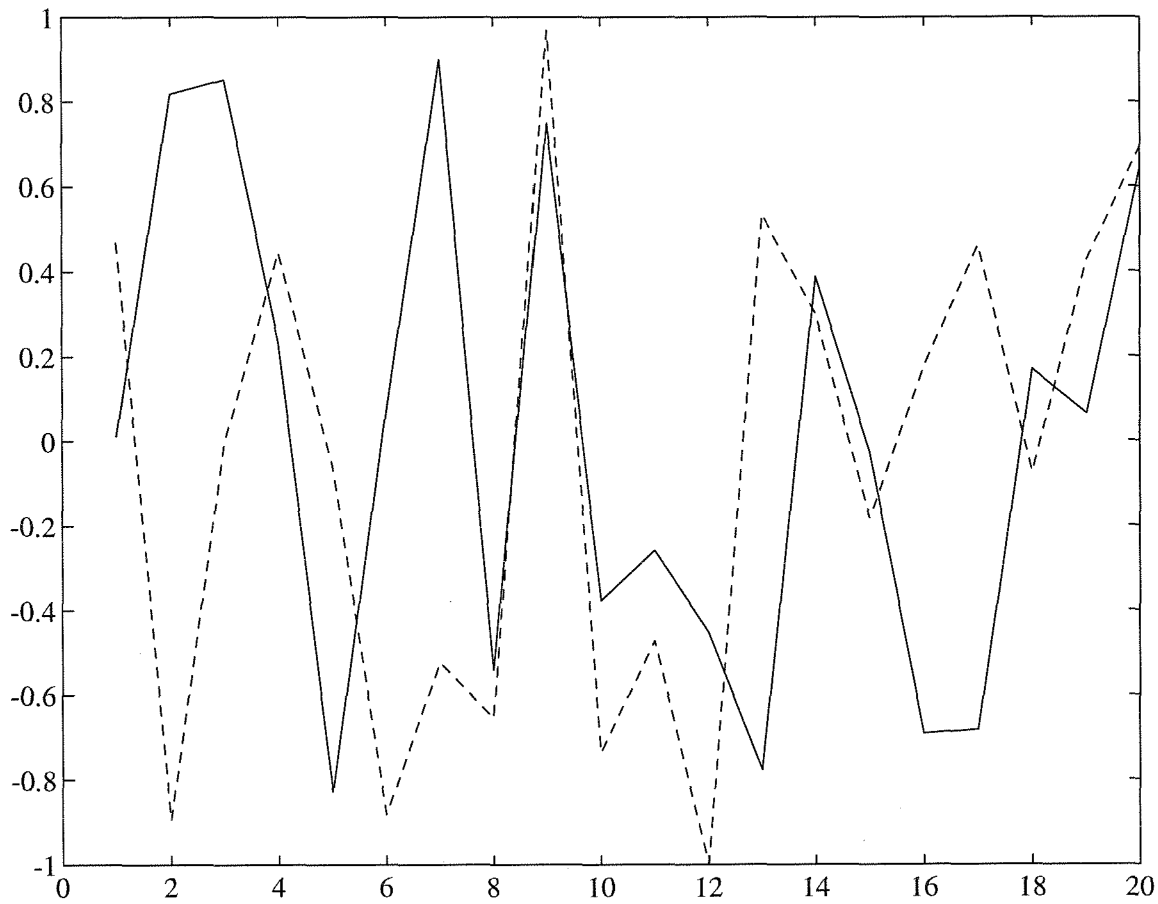

Figure 6.2: Time variations of parameters $\bar{\delta}_1$ (Solid) and $\bar{\delta}_2$ (dashed)

Another drawback of this algorithm is that the $\infty-$norm has to be used temporally. In many situations, we would prefer to use the $2-$norm (temporally) in the objective function. In the next chapter, we will propose a robust MPC algorithm for controlling uncertain linear time-invariant systems which overcomes these difficulties.

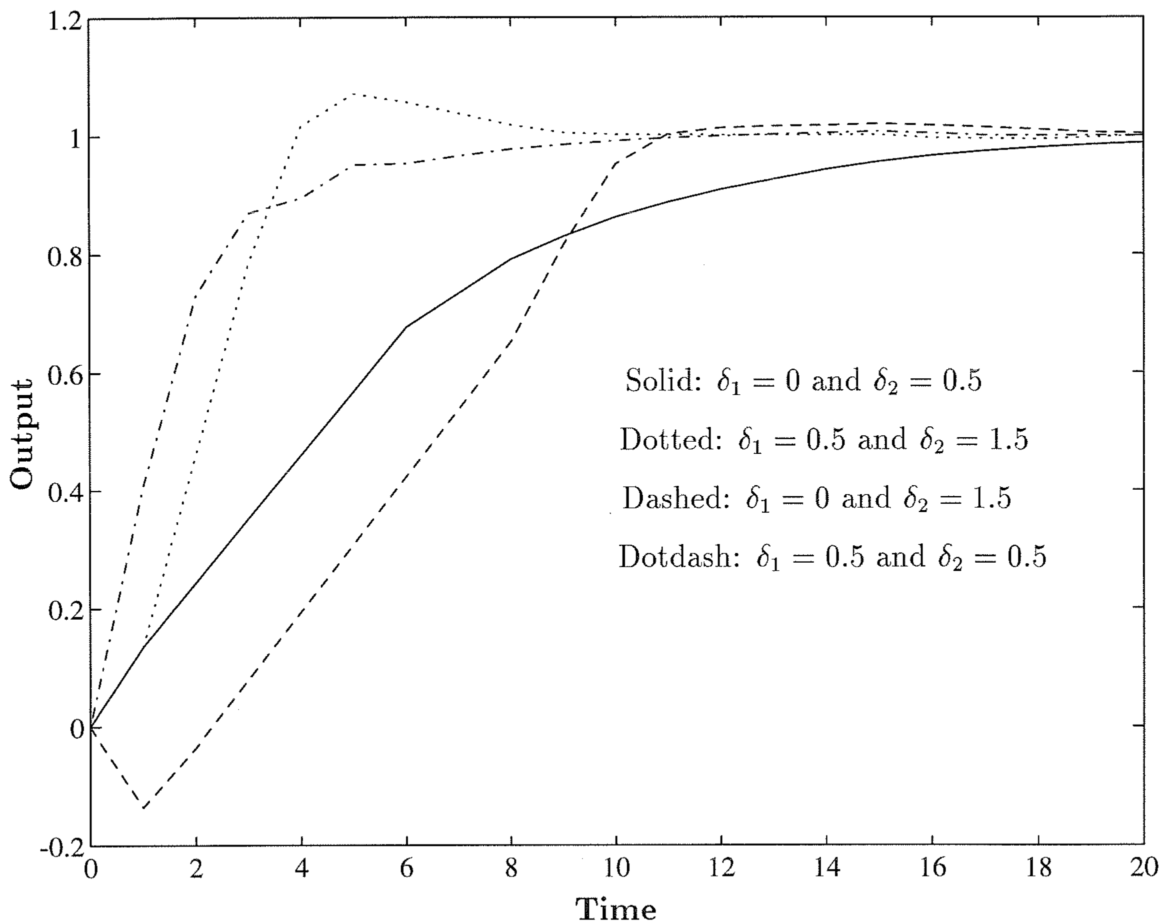

Figure 6.3: Responses for a set-point change

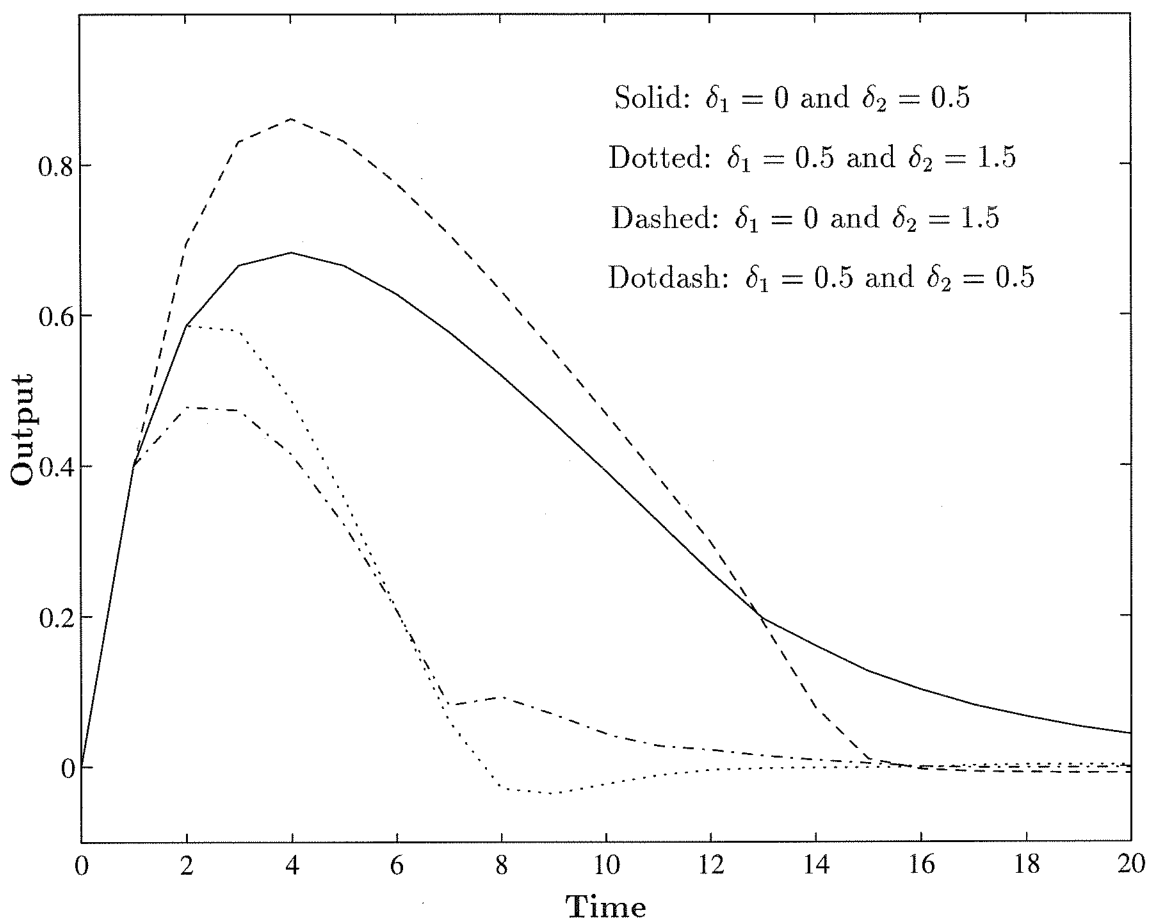

Figure 6.4: Disturbance rejection

Chapter 7 Robust Control of Linear Time Invariant Systems with Constraints

Summary

In this chapter, we propose a Model Predictive Control algorithm which optimizes performance subject to stability constraints for control of linear time invariant discrete-time systems with hard input constraints and soft output constraints. In the nominal case, we show that global asymptotic stability is guaranteed for both state feedback and output feedback. Furthermore, global asymptotic stability is preserved for all asymptotically constant disturbances.

The algorithm is then generalized to the robust case. We show that robust global asymptotic stability is guaranteed for a set of linear time-invariant stable systems. When the system is represented by a Finite Impulse Response model, we show that the optimization problem can be cast as a quadratic program of moderate size for a broad class of uncertainty descriptions.

7.1 Introduction

All real world control systems must deal with constraints. Although a rich theory has been developed for the robust control of linear systems, *very little* is known about the robust control of linear systems *with constraints*. In the previous chapter, we generalized the robust Mode Predictive Control (MPC) algorithm introduced by Campo and Morari [10] for control of uncertain linear *time-varying* (LTV) systems and proved several important results. In this chapter, we will consider linear *time-invariant* (LTI) systems.

Campo and Morari [10, 9] made the first rigorous attempt to extend the MPC concept to control of uncertain linear systems and proposed a robust MPC algorithm.

Unfortunately, it is well known that robust stability is not guaranteed with this algorithm. Zafiriou [96] used the contraction mapping principle to derive some necessary and some sufficient conditions for robust stability. However, the conditions are both conservative and difficult to verify. Assuming lower and upper bounds on each impulse response coefficient, Genceli and Nikolaos [32] showed how to determine weights such that robust stability can be guaranteed for a set of Finite Impulse Response (FIR) models. However, often weights do *not* exist even when robust stabilization is *possible* for a set of FIR models. Lee et al. [56] proposed a robust MPC algorithm that minimizes the expectation of a multi-step objective function for an input-output model with stochastic parameters. Of course, the concept of robust stability cannot be defined in this framework.

In the previous chapter, we generalized the robust MPC algorithm introduced by Campo and Morari [10] and demonstrated that this new robust MPC algorithm can robustly stabilize any set of linear systems represented by FIR models for which robust stabilization is possible. However, the controlled system had to be assumed to be *time-varying*. Thus applying this new robust MPC algorithm to a time-invariant system often produces conservative results. Another drawback of this robust MPC algorithm is that only the $\infty-$norm can be used temporally. This is because it is generally not possible to have a zero steady-state error for all plants in the set. Using the $1-$norm or $2-$norm instead of the $\infty-$norm temporally may result in an unbounded objective function because of the infinite output horizon. The focus of this chapter is to introduce an MPC algorithm which overcomes these difficulties.

This chapter is organized as follows. After presenting some preliminaries in Section 7.2, Section 7.3 deals with the nominal case. Specifically, a novel MPC algorithm which optimizes nominal performance subject to a nominal stability constraint for controlling LTI systems with "hard" input constraints and "soft" output constraints is proposed. With this scheme we then show that global asymptotic stability is guaranteed for both state feedback and output feedback. Furthermore, we show that global asymptotic stability is preserved for all asymptotically constant disturbances. The framework is generalized to handle the robust case in Section 7.4. We show that robust global asymptotic stability is guaranteed for a set of stable LTI systems. The output tracking problem is treated in Section 7.5. In Section 7.6, for the special case when the system is represented by an FIR model, we show that the optimization problems can be cast as quadratic programs of moderate size for a broad class of uncertainty descriptions. Several examples are presented in Section 7.7 to demonstrate characteristics of the proposed algorithm. Section 7.8 concludes the chapter.

Notations and Assumptions The notation used in this chapter is fairly standard. $|\bullet|$ denotes the Euclidean norm, $|x|_1$ the 1−norm, and $|x|_\infty$ the ∞−norm. $|x|_P = \sqrt{x^T P x}$ denotes the weighted Euclidean norm. x^T denotes the transpose of x. $||\bullet||$ denotes the induced 2−norm. Given two vectors x and y, $x \leq y \Leftrightarrow x_i \leq y_i \ \forall \ i$. Throughout the chapter, we assume that the plant is a stable LTI discrete-time system.

7.2 Preliminaries

Consider the following LTI system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + d(k) \end{aligned} \tag{7.1}$$

where $x(k) \in \Re^{n_x}$ is the state, $u(k) \in \Re^{n_u}$ the input, $y(k) \in \Re^{n_y}$ the output, and $d(k) \in \Re^{n_y}$ the disturbance. Denote the nominal model by (A_0, B_0, C_0) and the real plant by (A_p, B_p, C_p). The input is assumed to belong to the set $\mathcal{U}$ which is defined as follows.

$$\mathcal{U} \triangleq \{u : 0 > u^{min} \leq u \leq u^{max} > 0\} \tag{7.2}$$

The input constraints are always present and are imposed by physical limitations of the actuators which cannot be exceeded under any circumstances. Thus, $u(k) \in \mathcal{U} \ \forall \ k$. Often we may have bounds on the rate of changes in the input, i.e.

$$|\Delta u(k)| \leq \Delta u^{max}$$

We will also assume that the disturbance belongs to the set $\mathcal{D}$ which is defined as follows.

$$\mathcal{D} \triangleq \{d : |d|_\infty < \infty \text{ and } C(I-A)^{-1}Bu + d = 0 \text{ for some } u \in \mathcal{U}\} \tag{7.3}$$

Often it is desirable to keep specific outputs within certain limits for reasons related to plant operation, e.g. safety, material constraints, etc. Let

$$\mathcal{X} \triangleq \left\{ x : [F_x \;\; F_u] \begin{bmatrix} x \\ u \end{bmatrix} \leq f, u \in \mathcal{U} \right\} \tag{7.4}$$

It is usually unavoidable to exceed the output constraints, at least temporarily, for example, when the system is subject to unexpected disturbances. Thus, $x(k)$ does not necessarily belong to $\mathcal{X}$ for all k. However, we can relax the output constraints and assume that $x(k) \in \mathcal{X}_\epsilon \; \forall \; k$ defined as follows.

$$\mathcal{X}_\epsilon \triangleq \left\{ x : [F_x \;\; F_u] \begin{bmatrix} x \\ u \end{bmatrix} \leq f + \epsilon, \epsilon \geq 0, u \in \mathcal{U} \right\} \tag{7.5}$$

To make the control problem meaningful, we will assume the following:

- $u = 0$ is an *interior* point of $\mathcal{U}$.
- $\Delta u^{max} > 0$.
- $(x, u) = (0, 0)$ is an *interior* point of $\mathcal{X}$.

7.3 Nominal Stability

In this section, we assume that the plant is known, i.e. $A_p = A_0, B_p = B_0$, and $C_p = C_0$. An MPC algorithm that optimizes performance subject to a stability constraint is proposed. With this scheme we then show that closed loop asymptotic stability is guaranteed with both state feedback and output feedback. For all asymptotically constant disturbances, we show that asymptotic stability is preserved. We also remark that the stability constraint is necessary to ensure asymptotic stability in the unconstrained case.

7.3.1 State Feedback

Define the objective function as

$$\Phi_k(A,B,C) = \sum_{i=1}^{H_p} |x(k+i|k)|^2_{\Gamma_x} + \sum_{i=0}^{H_c} \left[|u(k+i|k)|^2_{\Gamma_u} + |\Delta u(k+i|k)|^2_{\Gamma_{\Delta u}} \right] + |\epsilon(k)|^2_{\Gamma_\epsilon} \tag{7.6}$$

where $\Gamma_x \geq 0, \Gamma_u > 0, \Gamma_{\Delta u} \geq 0, \Gamma_\epsilon \geq 0, H_p \geq H_c$, and H_c is finite. Γ_x, Γ_u, and $\Gamma_{\Delta u}$ are symmetric matrices. Γ_ϵ is a diagonal matrix. $(\bullet)(k+i|k)$ denotes the variable $(\bullet)$ at sampling time $k+i$ predicted (or calculated) at sampling time k. Define *Controller MPCC* as follows.

Definition 11 Controller MPCC: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \Phi_k(A_0, B_0, C_0)$$

subject to

$$\begin{cases} u(k+i|k) \in \mathcal{U} & i = 0,1,\cdots,H_c-1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0,1,\cdots,H_c-1 \\ \Delta u(k+i|k) = 0 & i = H_c, H_c+1, \cdots, \infty \\ x(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0,1,\cdots,\infty \end{cases} \tag{7.7}$$

and

$$|x(k+1|k)|_P \leq \lambda |x(k)|_P$$

where $P > 0, A^T PA - P = -Q,$[1] $Q > 0, \lambda < 1,$ *and* $\Phi_k(A_0, B_0, C_0)$ *is defined by (7.6).*

Remark 32 *For simplicity but without loss of generality, we have assumed here that we would like to stabilize the system to the origin. However, if we would like the state to go to some reference state, say* x_r*, then we need to replace* $|x(k+i|k)|_{\Gamma_x}$ *in the objective function by* $|x(k+i|k) - x_r|_{\Gamma_x}$ *and the contraction constraint* $|x(k+1|k)|_P \leq \lambda |x(k)|_P$ *by* $|x(k+1|k) - x_r|_P \leq \lambda |x(k) - x_r|_P$.

Remark 33 *We will show that the contraction constraint* $|x(k+1|k)|_P \leq \lambda |x(k)|_P, \lambda \in [0,1)$ *ensures that the closed loop system is asymptotically stable. Therefore, it is referred to as the "stability constraint" in the sequel.*

We want to show that global asymptotic stability is guaranteed with *Controller MPCC* when the state can be measured and there are no disturbances, i.e. $d(k) = 0 \ \forall \ k \geq 0$. Before we state the theorem on global stability, let us first prove the following lemma.

Lemma 9 *Suppose there are no disturbances, i.e.* $d(k) = 0 \ \forall \ k$*. Then there exists a constant* $\lambda^* \in [0,1)$ *such that the optimization problem (7.7) is feasible for all* $\lambda \in [\lambda^*, 1)$ *if* A *is stable.*

Proof. We want to show that $u(k+i|k) = 0, i = 0, \cdots, H_c - 1$, is a feasible solution for all $\lambda \in [\lambda^*, 1)$. Clearly $u(k+i|k) \in \mathcal{U}, i = 0, \cdots, H_c - 1$. Since A is stable, $x(k+i|k) \in \mathcal{X}_{\epsilon(k)} \ \forall \ i \geq 1$ for a sufficiently large finite $\epsilon(k)$. The existence of a constant $\lambda^* \in [0,1)$ can be shown as follows.

$$
\begin{aligned}
|x(k+1|k)|_P^2 &= |Ax(k) + Bu(k|k)|_P^2 \\
&= |Ax(k)|_P^2 \\
&= |x(k)|_{P-Q}^2 \\
&= |x(k)|_P^2 - |x(k)|_Q^2
\end{aligned}
$$

[1] $0 < P < \infty$ since A is assumed to be stable.

$$
\begin{aligned}
&\leq |x(k)|_P^2 - \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)}|x(k)|_P^2 \\
&= (1 - \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)})|x(k)|_P^2 \\
&\triangleq \lambda^{*^2}|x(k)|_P^2
\end{aligned}
$$

Obviously $\lambda^* \triangleq \sqrt{1 - \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)}} < 1$ for stable A and the stability constraint is feasible for all $\lambda \geq \lambda^*$. □

Remark 34 *In general, $\lambda^* < 1$ may not exist if other norms are used for the contraction constraint.*

Using Lemma 9, we can show the following theorem in the absence of the disturbance.

Theorem 23 (State Feedback) *Assume that A is stable and that $d(k) = 0 \; \forall \; k \geq 0$. Suppose the state is measured. For all $\lambda \in [\lambda^*, 1)$, where λ^* is defined as in Lemma 9, the closed-loop system with* Controller MPCC *is globally asymptotically stable.*

Proof. As shown in Lemma 9, a constant $\lambda^* \in [0, 1)$ exists such that $u(k+i|k) = 0, i = 0, \cdots, H_c - 1$, is feasible for all $\lambda \in [\lambda^*, 1)$ but may not be optimal. Thus,

$$
\begin{aligned}
J_k &\leq \sum_{i=1}^{H_p} |x(k+i|k)|_{\Gamma_x}^2 + |\epsilon(k)|_{\Gamma_\epsilon}^2 \\
&\leq \frac{\bar{\sigma}(\sum_{j=0}^{H_p}(A^j)^T \Gamma_x A^j)}{\underline{\sigma}(P)}|x(k)|_P^2 + |\epsilon(k)|_{\Gamma_\epsilon}^2
\end{aligned}
$$

Since A is stable, a finite constant γ clearly exists such that the output constraints are feasible with $\epsilon(k) = \gamma|x(k)|_\infty$. Then we have

$$
J_k \leq \bar{\gamma}|x(k)|_P^2
$$

where $\bar{\gamma}$ is a constant defined appropriately. Since $|x(k)|_P \leq \lambda|x(k-1)|_P \leq \lambda^k|x(0)|_P$ and $0 \leq \lambda < 1, x(k) \to 0$ and $J_k \to 0$ as $k \to \infty$. This together with $\Gamma_u > 0$ yields $x(k), u(k) \to 0$ as $k \to \infty$. □

Remark 35 *If we assume $\Gamma_u \geq 0$ and $\Gamma_{\Delta u} > 0$ instead of $\Gamma_u > 0$ and $\Gamma_{\Delta u} \geq 0$, Theorem 23 still holds. All we have to show is that the input is bounded in this case.*

Since $|x(k+1)|_P \leq \lambda |x(k)|_P$, $J_k \leq \gamma |x(k)|_P^2$ *where* $\gamma \triangleq \frac{\bar{\sigma}(\sum_{j=0}^{H_p}(A^j)^T \Gamma_x A^j)}{\underline{\sigma}(P)}$, *and* $\Gamma_{\Delta u} > 0$, $|\Delta u(k)| \leq \bar{\gamma} |x(k)|_P$ *where* $\bar{\gamma}$ *is defined appropriately. We have*

$$
\begin{aligned}
|u(k)| &= |u(0) + \sum_{i=1}^{k} \Delta u(i)| \\
&\leq |u(0)| + \sum_{i=1}^{k} |\Delta u(i)| \\
&\leq |u(0)| + \bar{\gamma} \sum_{i=1}^{k} \lambda^i |x(0)| \\
&= |u(0)| + \frac{1-\lambda^k}{1-\lambda} \bar{\gamma} |x(0)| < \infty \quad \forall\ k
\end{aligned}
$$

Therefore, the closed loop system is asymptotically stable.

Remark 36 *The stability constraint is sufficient to ensure stability. It is well known that for unconstrained linear systems, the closed loop system is stable if and only if the state matrix of the closed loop system is stable which is equivalent to the existence of a positive definite matrix* P *such that* $|x(k+1|k)|_P \leq \lambda |x(k)|_P$ *for some* $\lambda \in [0, 1)$. *Therefore, the stability constraint should not result in any conservatism in controller design.*

Remark 37 *With the stability constraint, the optimization problem (7.7) cannot be cast as a quadratic program. We can solve the optimization problem as follows:*

Step 1 *Solve the optimization problem without the stability constraint (quadratic program).*

Step 2 *Check if the stability constraint is satisfied. If yes, we are done; if no, add a penalty term of the form* $w|x(k+1|k)|_P^2$ *to the objective function or adjust the weight* w *and go to Step 1.*

Since the global optimal solution to the optimization problem (7.7) is not required for Theorem 23 to hold as long as the stability constraint is satisfied, we do not have to determine the optimal solution.

Remark 38 *As it can be seen from the proof, other objective functions (for example, those mentioned in Chapter 2) can be used. However, to keep the presentation simple and clear, we would stick with the objective function defined by (7.6).*

7.3.2 Output Feedback

In this section, we consider the case where the state has to be estimated. Since the closed loop system may be nonlinear because of the constraints, we cannot apply the Separation Principle to prove closed loop stability with output feedback. It is well known that, in general, a nonlinear closed loop system with the state estimated via an asymptotic observer can be unstable even though it is stable with state feedback. However, we want to show, for *Controller MPCC*, that closed loop stability is guaranteed when the state can be estimated with an asymptotic observer and when there are no disturbances, i.e. $d(k) = 0 \ \forall \ k \geq 0$.

Denote the state (output) at sampling time $k+i$ estimated at sampling time k by $\hat{x}(k+i|k)$ ($\hat{y}(k+i|k)$). The state is estimated as follows.

$$\begin{aligned} \hat{x}(k|k) &= A\hat{x}(k-1|k-1) + Bu(k-1) + K(y(k) - \hat{y}(k|k-1)) \\ \hat{x}(k+i|k) &= Ax(k+i-1|k) + Bu(k+i-1) \qquad i \geq 1 \end{aligned} \tag{7.8}$$

where K is the observer gain. Define *Output Feedback Controller MPCC* as follows.

Definition 12 Output Feedback Controller MPCC: *At sampling time k, the control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+$*

$1|k), \cdots, u(k+H_c-1|k)\}$ *which is the minimizer of the optimization problem*

$$\hat{J}_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \hat{\Phi}_k(A_0, B_0, C_0)$$

subject to

$$\begin{cases} u(k+i|k) \in \mathcal{U} & i = 0, 1, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c - 1 \\ \Delta u(k+i|k) = 0 & i = H_c, H_c + 1, \cdots, \infty \\ \hat{x}(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0, 1, \cdots, \infty \end{cases} \tag{7.9}$$

and

$$|\hat{x}(k+1|k)|_P \leq \lambda |\hat{x}(k)|_P$$

where

$$\hat{\Phi}_k(A, B, C) = \sum_{i=1}^{H_p} |\hat{x}(k+i|k)|^2_{\Gamma_x} + \sum_{i=0}^{H_c} \left[|u(k+i|k)|^2_{\Gamma_u} + |\Delta u(k+i|k)|^2_{\Gamma_{\Delta u}} \right] + |\epsilon(k)|^2_{\Gamma_\epsilon} \tag{7.10}$$

Here $P, \lambda, \Gamma_x, \Gamma_u$, *and* $\Gamma_{\Delta u}$ *are defined as in Definition 11.*

Combining this equation with equation (7.1) with $d(k) = 0 \; \forall \; k$ yields

$$e(k+1) = (I - KC)Ae(k) \tag{7.11}$$

where $e(k) = x(k) - \hat{x}(k|k)$ is the estimation error. Thus equation (7.8) can be written as

$$\begin{aligned} \hat{x}(k|k) &= \hat{x}(k|k-1) + KCAe(k-1) \\ \hat{x}(k+i|k) &= A\hat{x}(k+i-1|k) + Bu(k+i-1|k) \qquad i \geq 1 \end{aligned} \tag{7.12}$$

We have the following lemma for *Output Feedback Control MPCC*. The proof is omitted since similar arguments in proving Lemma 9 can be used here.

Lemma 10 *Suppose there are no disturbances, i.e.* $d(k) = 0 \; \forall \; k$. *Then there exists* *a constant* $\lambda^* \in [0, 1)$ *such that the optimization problem (7.9) is feasible for all* $\lambda \in [\lambda^*, 1)$ *if* A *is stable.*

Before we state the theorem on stability, let us first prove the following lemma.

Lemma 11 *Assume that* A *and* $(I - KC)A$ *are stable, i.e. all the eigenvalues are strictly inside the unit circle. Suppose* $d(k) = 0 \ \forall \ k \geq 0$. *Then* $\sum_{i=1}^{k} \lambda^{i-1} |KCAe(k - i)|_P \to 0$ *as* $k \to \infty$ *for all* $\lambda \in [\lambda^*, 1)$.

Proof. From equation (7.11), we have

$$|e(k)|_P \leq ck^{\alpha-1}\rho^k |e(0)|_P$$

where $\rho = \lambda_{max}((I - KC)A)$, c is a constant and α is the multiplicity associated with the eigenvalue for the spectral radius of $(I - KC)A$. Here $\lambda_{max}(A)$ denotes the spectral radius of A. Stability of A insures the existence of $\lambda^* \in [0, 1)$ and stability of $(I - KC)A$ implies $\rho < 1$. Thus,

$$\begin{aligned}
\sum_{i=1}^{k} \lambda^{i-1} |KCAe(k-i)|_P &\leq \bar{\sigma}(P^{\frac{1}{2}} KCAP^{-\frac{1}{2}}) \sum_{i=1}^{k} \lambda^{i-1} |e(k-i)|_P \\
&\leq \bar{\sigma}(P^{\frac{1}{2}} KCAP^{-\frac{1}{2}}) \sum_{i=1}^{k} \lambda^{i-1} c(k-i)^{\alpha-1} \rho^{k-i} |e(0)|_P \\
&= c\bar{\sigma}(P^{\frac{1}{2}} KCAP^{-\frac{1}{2}}) |e(0)|_P \sum_{i=1}^{k} \lambda^{i-1} (k-i)^{\alpha-1} \rho^{k-i} \\
&\leq c\bar{\sigma}(P^{\frac{1}{2}} KCAP^{-\frac{1}{2}}) |e(0)|_P \max(\lambda, \rho)^{k-1} \sum_{i=1}^{k} (k-i)^{\alpha-1} \\
&\leq c\bar{\sigma}(P^{\frac{1}{2}} KCAP^{-\frac{1}{2}}) |e(0)|_P \max(\lambda, \rho)^{k-1} \sum_{i=1}^{k} k^{\alpha-1} \\
&\leq c\bar{\sigma}(P^{\frac{1}{2}} KCAP^{-\frac{1}{2}}) |e(0)|_P \max(\lambda, \rho)^{k-1} k^{\alpha}
\end{aligned}$$

Since $0 \leq \max(\lambda, \rho) < 1$, $\max(\lambda, \rho)^{k-1} k^{\alpha}$ approaches zero as $k \to \infty$ and we have the desired result. □

The following theorem states that global asymptotic stability with output feedback can be guaranteed for stable systems.

Theorem 24 (Output Feedback) *Assume that A and $(I-KC)A$ are stable, i.e. all eigenvalues of A and $(I-KC)A$ are strictly inside the unit circle. Suppose $d(k)=0\ \forall\ k\geq 0$. Then the overall system with* Output Feedback Controller MPCC *is globally asymptotically stable for all $\lambda\in[\lambda^*,1)$ where $\lambda^*\triangleq\sqrt{1-\frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)}}$.*

Proof. As shown in Lemma 9, $u(k+i|k)=0, i=0,\cdots,H_c-1$, is feasible for all $\lambda\in[\lambda^*,1)$ but may not be optimal. Thus,

$$\begin{aligned}J_k &\leq \sum_{i=1}^{H_p}|\hat{x}(k+i|k)|^2_{\Gamma_x}+|\epsilon(k)|^2_{\Gamma_\epsilon}\\ &\leq \frac{\bar{\sigma}(\sum_{i=1}^{H_p}(A^i)^T\Gamma_x A^i)}{\underline{\sigma}(P)}|\hat{x}(k|k)|^2_P+|\epsilon(k)|^2_{\Gamma_\epsilon}\end{aligned}$$

Following similar arguments as in the proof of Theorem 23, we have

$$J_k\leq\gamma|\hat{x}(k|k)|^2_P$$

Now, we want to show that $J_k\to 0$ as $k\to\infty$. From Equation (7.12), we have

$$\hat{x}(k|k)=\hat{x}(k|k-1)+KCAe(k-1)$$

Thus,

$$\begin{aligned}|\hat{x}(k|k)|_P &= |\hat{x}(k|k-1)+KCAe(k-1)|_P\\ &\leq |\hat{x}(k|k-1)|_P+|KCAe(k-1)|_P\\ &\leq \lambda|\hat{x}(k-1|k-1)|_P+|KCAe(k-1)|_P\\ &\leq \lambda^k|\hat{x}(0)|_P+\sum_{i=1}^{k}\lambda^{i-1}|KCAe(k-i)|_P\end{aligned}$$

Since $0\leq\lambda<1$, the first term clearly approaches zero as $k\to\infty$. By Lemma 11, the second term approaches zero as $k\to\infty$ for $\lambda\in[\lambda^*,1)$. Therefore, $\hat{x}(k|k)\to 0$ as $k\to\infty$ which in turn yields $x(k)$ and $J_k\to 0$ as $k\to\infty$. This together with $\Gamma_u>0$ yields $u(k)\to 0$ asymptotically. $\square$

Remark 39 *By similar arguments as in Remark 35, we can show Theorem 24 holds for* $\Gamma_u \geq 0$ *and* $\Gamma_{\Delta u} > 0$ *as well.*

7.3.3 Disturbance Rejection

In this section, we investigate how disturbances affect closed loop stability. We show that, with a modified stability constraint, global asymptotic stability is preserved with output feedback (hence state feedback) for asymptotically constant disturbances. Consider the following extended system:

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ d(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta d(k) \\ y(k) &= [C \ I] \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \end{aligned} \tag{7.13}$$

where $\Delta d(k) = d(k+1) - d(k)$. Let

$$\bar{x}(k) = \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \qquad \bar{A} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \qquad \bar{C} = [C \ \ I]$$

Both the state and disturbance can be estimated as follows.

$$\begin{aligned} \hat{\bar{x}}(k|k) &= \bar{A}\hat{\bar{x}}(k-1|k-1) + \bar{B}u(k-1) + \bar{K}(y(k) - \hat{y}(k|k-1)) \\ \hat{\bar{x}}(k+i|k) &= \bar{A}\hat{\bar{x}}(k+i-1|k) + \bar{B}u(k+i-1) \qquad i \geq 1 \end{aligned} \tag{7.14}$$

where $\bar{K}$ is the observer gain. Define *Output Feedback Regulator MPCC* as follows.

Definition 13 Output Feedback Regulator MPCC: *At sampling time* k*, the control move* $u(k)$ *equals the first element* $u(k|k)$ *of the sequence* $\{u(k|k), u(k+$

$1|k), \cdots, u(k+H_c-1|k)\}$ *which is the minimizer of the optimization problem*

$$\hat{J}_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \hat{\Phi}_k^R$$

subject to

$$\begin{cases} u(k+i|k) \in \mathcal{U} & i = 0,1,\cdots,H_c-1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0,1,\cdots,H_c-1 \\ \Delta u(k+i|k) = 0 & i = H_c, H_c+1,\cdots,\infty \\ \hat{x}(k+i|k) \in \mathcal{X}_{\epsilon(k)} & i = 0,1,\cdots,\infty \end{cases}$$

and

$$|\hat{x}(k+1|k) - \hat{x}^{ss}(k)|_P \leq \lambda|\hat{x}(k|k) - \hat{x}^{ss}(k)|_P + \min_{v\in\mathcal{U}} |A\hat{x}^{ss} + Bv - \hat{x}^{ss}(k)|_P \tag{7.15}$$

where $\hat{x}(k+i|k)$ *and* $d(k|k)$ *are estimated via Equation (7.14),* $\hat{x}^{ss}(k) = (I-A)^{-1}Bv$*, where* v *is such that* $C(I-A)^{-1}Bv + \hat{d}(k) = 0$*,*[2] *is the estimate of the steady-state values at sampling time* k*, and*

$$\hat{\Phi}_k^R = \sum_{i=1}^{H_p} |y(k+i|k)|^2_{\Gamma_y} + \sum_{i=0}^{H_c} \left[|u(k+i|k)|^2_{\Gamma_u} + |\Delta u(k+i|k)|^2_{\Gamma_{\Delta u}}\right] + |\epsilon(k)|^2_{\Gamma_\epsilon}$$

with $\Gamma_y \geq 0$*. Here* P, λ, Γ_u*, and* $\Gamma_{\Delta u}$ *are defined as in Definition 11.*

Combining the observer equation (7.14) with equation (7.13) yields

$$\bar{e}(k+1) = (I - \bar{K}\bar{C})\bar{A}\bar{e}(k) + \bar{K}\Delta d(k) \tag{7.16}$$

[2]Clearly v may not be unique. If this is the case, we can determine v such that $v^T v$ is minimized.

where $\bar{e}(k) = \bar{x}(k) - \hat{\bar{x}}(k|k)$. Since A is stable, an asymptotic observer exists, i.e. $\bar{K}$ exists such that $(I - \bar{K}\bar{C})\bar{A}$ is stable. Equivalently equation (7.14) can be written as

$$\begin{aligned} \hat{\bar{x}}(k|k) &= \hat{\bar{x}}(k|k-1) + \bar{K}\bar{C}\bar{A}\bar{e}(k-1) + \bar{K}\Delta d(k-1) \\ \hat{\bar{x}}(k+i|k) &= \bar{A}\hat{x}(k+i-1|k) + \bar{B}u(k+i-1|k) \qquad i \geq 1 \end{aligned} \tag{7.17}$$

With these preliminaries, we can prove the following theorem.

Theorem 25 (Output Feedback) *Assume that A is stable, that $\bar{K}$ is such that $(I - \bar{K}\bar{C})\bar{A}$ is stable, and that $d(k) \in \mathcal{D}\ \forall\ k \geq 0$.*[3] *Then with* Output Feedback Regulator MPCC *the optimization problem (7.15) is feasible for all $k \geq 0$ and $\lambda \in [\lambda^*, 1)$ where λ^* is defined as in Lemma 9. Furthermore, the closed-loop system is globally asymptotically stable for all asymptotically constant disturbances.*

Proof. $u(k|k) = \arg\min_{v\in\mathcal{U}} |A\hat{x}^{ss} + Bv - \hat{x}^{ss}(k)|_P \in \mathcal{U}$ and $x(k+i|k) \in \mathcal{X}_{\epsilon(k)}\ \forall\ i \geq 1$ for sufficiently large $\epsilon(k)$. That the stability constraint is feasible for $u(k|k) = \arg\min_{v\in\mathcal{U}} |A\hat{x}^{ss} + Bv - \hat{x}^{ss}(k)|_P$ can be shown as follows.

$$\begin{aligned} |\hat{x}(k+1|k) - \hat{x}^{ss}(k)|_P &= |A\hat{x}(k|k) + Bu(k) - \hat{x}^{ss}(k)|_P \\ &= |A\hat{x}(k|k) - A\hat{x}^{ss}(k) + A\hat{x}^{ss}(k) + Bu(k) - \hat{x}^{ss}(k)|_P \\ &\leq |A(\hat{x}(k|k) - \hat{x}^{ss}(k))|_P + |A\hat{x}^{ss}(k) + Bu(k) - \hat{x}^{ss}(k)|_P \\ &\leq \lambda^*|\hat{x}(k|k) - \hat{x}^{ss}(k)|_P + |A\hat{x}^{ss}(k) + Bu(k) - \hat{x}^{ss}(k)|_P \\ &= \lambda^*|\hat{x}(k|k) - \hat{x}^{ss}(k)|_P + \min_{v\in\mathcal{U}} |A\hat{x}^{ss}(k) + Bv - \hat{x}^{ss}|_P \end{aligned}$$

Thus the optimization problem (7.7) is feasible for all $\lambda \in [\lambda^*, 1)$ and $k \geq 0$. Global asymptotic stability of the closed loop system can be shown as follows.

$$\begin{aligned} |\hat{x}(k|k) - \hat{x}^{ss}(k)|_P &= |\hat{x}(k|k) - \hat{x}^{ss}(k-1)|_P + |\hat{x}^{ss}(k) - \hat{x}^{ss}(k-1)|_P \\ &= |\hat{x}(k|k-1) + \bar{K}\bar{C}\bar{A}\bar{e}(k-1) + \bar{K}\Delta d(k-1) - \hat{x}^{ss}(k-1)|_P \\ &\quad + |\hat{x}^{ss}(k) - \hat{x}^{ss}(k-1)|_P \end{aligned}$$

[3] The theorem holds as well if there exists a finite T such that $d(k) \in \mathcal{D}\ \forall\ k \geq T$.

$$
\begin{aligned}
&\leq |\hat{x}(k|k-1) - \hat{x}^{ss}(k-1)|_P + |\bar{K}\bar{C}\bar{A}\bar{e}(k-1) + \bar{K}\Delta d(k-1)|_P \\
&\quad + |\hat{x}^{ss}(k) - \hat{x}^{ss}(k-1)|_P \\
&\leq \lambda|\hat{x}(k|k) - \hat{x}^{ss}(k)|_P + \min_{v\in\mathcal{U}} |A\hat{x}^{ss}(k) + Bv - \hat{x}^{ss}(k)|_P \\
&\quad + |\bar{K}\bar{C}\bar{A}\bar{e}(k-1) + \bar{K}\Delta d(k-1)|_P + |\hat{x}^{ss}(k) - \hat{x}^{ss}(k-1)|_P \\
&\triangleq \lambda|\hat{x}(k-1|k-1) - \hat{x}^{ss}(k-1)|_P + \xi(k-1)
\end{aligned}
$$

which yields

$$
|\hat{x}(k|k) - \hat{x}^{ss}(k)|_P \leq \lambda^k |\hat{x}(0) - \hat{x}^{ss}(0)|_P + \sum_{i=0}^{k} \lambda^{k-i}\xi(i)
$$

For asymptotically constant disturbances, $\Delta d(k) \to 0$ asymptotically. By Equation (7.16) and stability of $(I - \bar{K}\bar{C})\bar{A}$, $\bar{e}(k) \to 0$ and therefore $x^{ss}(k) - x^{ss}(k-1) \to 0$ asymptotically. This together with the assumption $d(k) \in \mathcal{D}\ \forall\ k$ implies that $\min_{v\in\mathcal{U}} |A\hat{x}^{ss}(k) + Bv - \hat{x}^{ss}(k)|_P$ either becomes zero after some finite time or approaches zero asymptotically. We have $\xi(k) \to 0$ asymptotically. Therefore, $\hat{x}(k) \to \hat{x}^{ss}(k)$ asymptotically which in turn implies $y(k) \to 0$ asymptotically. Since the objective function is bounded and $\Gamma_u > 0$, $u(k)$ is bounded. □

Remark 40 *By similar arguments as in Remark 35, we can show Theorem 25 holds for* $\Gamma_u \geq 0$ *and* $\Gamma_{\Delta u} > 0$ *as well.*

7.4 Robust Stability

Consider the LTI system (7.1) and assume that $d(k) = 0\ \forall\ k \geq 0$. The actual plant, (A_p, B_p, C_p), is not known exactly and is assumed to lie in some set, i.e. $(A_p, B_p, C_p) \in (\mathcal{A}, \mathcal{B}, \mathcal{C})$. At this point, the set can be completely arbitrary. The goal is to design an MPC controller such that closed loop stability is guaranteed for all plants in the set. Define *Robust Controller MPCC* as follows:

Definition 14 Robust Controller MPCC:

Step 0 Input the data.

Step 1 Set $k_0 = k$ and $i = 1$ where k denotes the current sampling time.

Step 2 The current control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+H_c-1|k)\}$ which is the minimizer of the optimization problem

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \Phi_k(A_0, B_0, C_0)$$

subject to

$$\begin{cases} u(k+j|k) \in \mathcal{U} & j = 0, 1, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c - 1 \\ \Delta u(k+j|k) = 0 & j = H_c, H_c + 1, \cdots, \infty \\ x(k+j|k) \in \mathcal{X}_\epsilon & j = 0, 1, \cdots, \infty \end{cases} \tag{7.18}$$

and the robust stability constraint

$$\sup_{(A,B)} \left| A^L x(k_0) + \mathcal{C}_L U(k_0|i) \right|_{\hat{P}} \leq \lambda |x(k_0)|_{\hat{P}} \tag{7.19}$$

where $\Phi_k(A_0, B_0, C_0)$ is defined by (7.6) and

$$\begin{aligned} \lambda &< 1 \\ \mathcal{C}_L &= [A^{L-1}B \;\; A^{L-2}B \;\; \cdots \;\; B] \\ U(k_0|i) &= \begin{bmatrix} u(k_0) \\ \vdots \\ u(k_0+i-2) \\ u(k_0+i-1|k_0+i-1) \\ \vdots \\ u(k_0+L-1|k_0+i-1) \end{bmatrix} \\ \hat{P} &> 0 \text{ is a weighting matrix} \end{aligned}$$

Step 3 Set $k = k + 1$. If $i = L$ or $|x(k_0 + i)|_{\hat{P}} \leq \lambda |x(k_0)|_{\hat{P}}$, go to Step 1; otherwise, set $i = i + 1$ and go to Step 2.

Remark 41 Robust Controller MPCC *optimizes nominal performance subject to a robust stability constraint. Clearly, other objectives can also be used. For example, if* $\max_{(A,B,C)} \Phi_k(A, B, C)$ *is minimized instead of* $\Phi_k(A_0, B_0, C_0)$, *then* Robust Controller MPCC *would optimize the worst-case performance subject to a robust stability constraint. However, as we shall see later, optimizing nominal performance subject to a robust stability constraint greatly simplifies computations.*

Lemma 12 *Assume that A is stable for all $A \in \mathcal{A}$. Then there exist an integer L and a constant $\lambda^*(L, \hat{P}) \in [0, 1)$ such that the optimization problem (7.18) is feasible for all $\lambda \in [\lambda^*(L, \hat{P}), 1)$.*

Proof. It suffices to prove the lemma for $i = 1$ where i is defined as in Step 2 of *Robust Controller MPCC*: if $i \neq 1$, then $u(k_0 + i + j|k_0 + i) = u(k_0 + i + j|k_0 + i - 1), j = 0, \cdots, H_c - 1$, is clearly a feasible solution. Let $u(k_0 + j|k_0) = 0, j = 0, \cdots, H_c - 1$. Clearly, $u(k_0 + j|k_0) = 0, j = 0, \cdots, H_c - 1$, may not be optimal but we want to show that it is a feasible solution. $u(k + j|k) \in \mathcal{U}, j = 0, \cdots, H_c - 1$. Since A is stable for all $A \in \mathcal{A}$, choosing $\epsilon(k)$ sufficiently large guarantees $x(k + j|k) \in \mathcal{X}_{\epsilon(k)}\ \forall\ j \geq 1$. The existence of an integer L and a constant $\lambda^*(L, \hat{P}) \in [0, 1)$ such that the robust stability constraint is satisfied for all $\lambda \in [\lambda^*(L, \hat{P}), 1)$ can be shown as follows:

$$
\begin{aligned}
\max_{(A,B)} \left| A^L x(k_0) + \mathcal{C}_L U(k_0|0) \right|_{\hat{P}} &= \max_A |A^L x(k_0)|_{\hat{P}} \\
&\leq \max_A \sqrt{\bar{\sigma}(\hat{P}^{\frac{1}{2}} A^L \hat{P}^{-\frac{1}{2}})} |x(k_0)|_{\hat{P}} \\
&\triangleq \lambda^*(L, \hat{P}) |x(k_0)|_{\hat{P}}
\end{aligned}
$$

Since A is stable for all $A \in \mathcal{A}$, a finite integer L exists such that $\lambda^*(L, \hat{P}) \in [0, 1)$. □

Remark 42 *Here the 2−norm is used for the robust stability constraint. Lemma 12 holds as well if other norms (e.g. the 1−norm) are used. As we shall see later, use of other norms may simplify computations.*

The following theorem states that global asymptotic robust stability is guaranteed with state feedback.

Theorem 26 (Robust State Feedback) *Assume that A is stable for all $A \in \mathcal{A}$ and that L is such that a constant $\lambda^*(L,\hat{P}) \in [0,1)$ exists. For all $\lambda \in [\lambda^*(L,\hat{P}),1)$, the closed-loop system with state feedback is globally asymptotically stable with* Robust Controller MPCC *for all $(A,B) \in (\mathcal{A},\mathcal{B})$.*

Proof. By Lemma 12, $\lambda^*(L,\hat{P}) \in [0,1)$ exists and the feasibility of the optimization problem is guaranteed for all $\lambda \in [\lambda^*(L,\hat{P}),1)$. By the robust stability constraint, we have

$$\begin{aligned}
|x(L(j+1))|_{\hat{P}} &\leq \max_{(A,B)} \left| A^L x(Lj) + \mathcal{C}_L \begin{bmatrix} u(Lj) \\ u(Lj+1) \\ \vdots \\ u(L(j+1)-1) \end{bmatrix} \right|_{\hat{P}} \\
&\leq \lambda |x(Lj)|_{\hat{P}} \\
&\leq \lambda^{j+1} |x(0)|_{\hat{P}}
\end{aligned}$$

Thus, $x(k) \to 0$ as $k \to \infty$. Following a similar argument as in Theorem 23, we can conclude that $u(k) \to 0$ as $k \to \infty$. □

Remark 43 *The robust stability constraint is sufficient to ensure robust stability. It is also necessary in the following sense: Given any controller, if there does not exist a positive definite matrix P such that $\sup_A \sqrt{\bar{\sigma}(\hat{P}^{\frac{1}{2}} A^L \hat{P}^{-\frac{1}{2}})}|x(k_0)|_{\hat{P}} \leq \lambda |x(k)|_P$ for some $\lambda \in [0,1)$ and for some integer L, then $x \not\to 0$ as $k \to \infty$ for some plant. Thus asymptotic stability is not guaranteed.*

The optimization problem (7.18) can be computationally expensive to solve for general uncertainty descriptions because of the robust stability constraint (7.19). In Section 7.6, for systems represented by FIR models, we will show that (7.19) can be represented by a set of linear constraints for a broad class of uncertainty descriptions. Thus the optimization problem (7.18) can be cast as a quadratic problem.

7.5 Output Tracking

So far we have been concerned with global stabilization to the origin. In this section, we deal with the constant output tracking problem. Because of the input constraints, tracking of an arbitrary constant output may not be possible. Let us define the set for which offset-free tracking may be possible.

$$\mathcal{Y} \triangleq \{y : \forall\,(A, B, C) \in (\mathcal{A}, \mathcal{B}, \mathcal{C}),\ \exists\, u \in \mathcal{U} \text{ such that } y - C(I-A)^{-1}Bu = 0\} \quad (7.20)$$

Clearly, the origin belongs to $\mathcal{Y}$.[4] Since the system is stable, integral control is necessary to obtain offset-free tracking. It is well known that robust integral control may not be possible for some uncertainty set [66]. Consider, for example, a set of SISO plants with both positive and negative steady-state gains. Then integral control with robust stability guarantee is *not* possible.

Let us define the objective function for output tracking.

$$\Phi_k^T(A, B, C) = \sum_{i=1}^{H_p} |r - y(k+i|k)|^2_{\Gamma_y} + \sum_{i=0}^{H_c} (|u(k+i|k)|^2_{\Gamma_u} + |\Delta u(k+i|k)|^2_{\Gamma_{\Delta u}}) + |\epsilon(k)|^2_{\Gamma_\epsilon} \quad (7.21)$$

where $\Gamma_y \geq 0, \Gamma_u \geq 0, \Gamma_{\Delta u} > 0$, and $\Gamma_\epsilon \geq 0$. Again Γ_ϵ is diagonal.

In the case of global stabilization to the *origin*, doing nothing, i.e. no control action, will steer both the state and the output to the origin for stable systems. However, in the case of output tracking, nonzero control action is *necessary* to guarantee offset-free tracking. Let us write the system (7.1) in difference form with $d(k) = 0\ \forall\, k$.

$$\begin{aligned} \Delta x(k+1) &= A\Delta x(k) + B\Delta u(k) \\ \Delta y(k) &= C\Delta x(k) \end{aligned} \quad (7.22)$$

[4]The origin may not be an interior point of $\mathcal{Y}$.

where $\Delta x(k) = x(k) - x(k-1)$, and $\Delta u(k)$ and $\Delta y(k)$ are defined similarly.

Consider

$$\max_{y^{ss}(k_0+L)\in Y^{ss}(k_0+L)} |r - y^{ss}(k_0+L)| \leq \lambda \max_{y^{ss}(k_0)\in Y^{ss}(k_0)} |r - y^{ss}(k_0)| \tag{7.23}$$

where r denotes the setpoint and $y^{ss}(k_0)$ denotes the steady-state output assuming that the control input remains constant after sampling time k_0 and the sets $Y^{ss}(k_0+L)$ and $Y^{ss}(k_0)$ are defined as follows.

$$Y^{ss}(j) = \left\{ y^{ss}(j) : y^{ss}(j) = y(j) + C\sum_{i=1}^{\infty} A^i \Delta x(j), (A, C) \in (\mathcal{A}, \mathcal{C}) \right\}, j = k_0, k_0 + L$$

Then for all $\lambda \in [0, 1)$, the output approaches the setpoint asymptotically. With appropriate assumptions, it can be easily shown that the input is also bounded. This together with stability of A gives that the closed loop system is globally asymptotically stable. Unfortunately, constraint (7.23) may be *infeasible*. This can be seen as follows for $\Delta u(k_0 + i) = 0\ \forall\ i \geq 0$.

$$\begin{aligned} y^{ss}(k_0+L) &= y(k_0+L) + C\sum_{i=1}^{\infty} A^i \Delta x(k_0+L) \\ &= y(k_0) + C_p \sum_{i=1}^{L} A_p^i \Delta x(k_0) + C\sum_{i=1}^{\infty} A^i A_p^L \Delta x(k_0) \\ &= y(k_0) + C_p \sum_{i=1}^{\infty} A_p^i \Delta x(k_0) + \left[C\sum_{i=1}^{\infty} A^i - C_p \sum_{i=1}^{\infty} A_p^i \right] A_p^L \Delta x(k_0) \\ &= y(k_0) + C_p \sum_{i=1}^{\infty} A_p^i \Delta x(k_0) + M_1 \Delta x(k_0) \end{aligned}$$

which yields,

$$\max_{(A,C),(A_p,C_p)} |r - y^{ss}(k_0+L)| \geq \max_{y^{ss}(k_0)\in Y^{ss}(k_0)} |r - y^{ss}(k_0)| + \gamma |\Delta x(k_0)|, \qquad \gamma \geq 0$$

For sufficiently large $\Delta x(k_0)$, we can show that (7.23) is not feasible for any nonzero bounded control moves if $M_1 \neq 0$. Therefore, unless $M_1 = 0$, constraint (7.23) may be infeasible for some $\Delta x(k_0)$. This motivates the following constraint.

$$\max_{y^{ss}(k_0+L|k_0)\in Y^{ss}(k_0+L|k_0)} |r - y^{ss}(k_0+L|k_0)|_{\hat{P}_2} \leq \lambda_2|\Delta x(k_0)|_{\hat{P}_1} + \beta_2(k_0) \max_{y^{ss}(k)\in Y^{ss}(k)} |r - y^{ss}(k_0)|_{\hat{P}_2} \tag{7.24}$$

where λ_2 and $\beta_2(k_0)$ are positive constant, $\hat{P}_1$ and $\hat{P}_2$ are positive definite weighting matrices, and

$$\begin{aligned} Y^{ss}(k_0+L|k_0) \quad &= \{y^{ss}(k_0+L|k_0) : y^{ss}(k_0+L|k_0) = y(k_0) + \\ &+ \left[\tilde{C}\sum_{i=0}^{L-1}\tilde{A}^i + C\sum_{i=1}^{\infty}A^i\tilde{A}^{L-1} \quad \cdots \quad \tilde{C} + C\sum_{i=1}^{\infty}A^i\right]\tilde{B}\Delta U(k_0|i), \\ &\left(\tilde{C}\sum_{i=1}^{L}\tilde{A}^i + C\sum_{i=1}^{\infty}A^i\tilde{A}^L\right)\Delta x(k_0), (A,B,C), (\tilde{A},\tilde{B},\tilde{C}) \in (\mathcal{A},\mathcal{B},\mathcal{C})\} \end{aligned}$$

This constraint alone does not, however, ensure global asymptotic stability because of the first term on the right-hand-side. We need to introduce the following additional constraint to ensure global asymptotic stability.

$$\max_{(A,B)\in(\mathcal{A},\mathcal{B})} |\Delta x(k_0+L|k_0)|_{\hat{P}_1} \leq \lambda_1|\Delta x(k_0)|_{\hat{P}_1} + \beta_1(k_0) \max_{y^{ss}(k)\in Y^{ss}(k)} |r - y^{ss}(k_0)|_{\hat{P}_2} \tag{7.25}$$

where λ_1 and $\beta_1(k_0)$ are positive scalars, and

$$\Delta x(k_0+L|k_0) = A^L\Delta x(k_0) + \sum_{j=0}^{L-1} A^{L-1-j}B\Delta u(k_0+j)$$

With these preliminaries, we state the control algorithm for output tracking.

Definition 15 Robust Tracking Controller MPCC :

Step 0 Input the data.

Step 1 Set $k_0 = k$ and $i = 1$ where k denotes the current sampling time.

Step 2 The current control move $u(k)$ equals the first element $u(k|k)$ of the sequence $\{u(k|k), u(k+1|k), \cdots, u(k+L-1|k)\}$ which is the minimizer of the optimization *problem*

$$J_k = \min_{u(k|k),\cdots,u(k+H_c-1|k),\epsilon(k)} \Phi_k^T(A_0, B_0, C_0)$$

subject to

$$\begin{cases} u(k+j|k) \in \mathcal{U} & j = 0, 1, \cdots, H_c - 1 \\ |\Delta u(k+i|k)| \leq \Delta u^{max} & i = 0, 1, \cdots, H_c - 1 \\ \Delta u(k+j|k) = 0 & j = H_c, H_c + 1, \cdots, \infty \\ x(k+j|k) \in \mathcal{X}_{\epsilon(k)} & j = 0, 1, \cdots, \infty \end{cases} \tag{7.26}$$

and constraints (7.24) and (7.25)

where $\Phi_k^T(A_0, B_0, C_0)$ is defined by (7.21).

Step 3 Set $k = k_0 + i$. If $i = L$ or

$$\begin{cases} |\Delta x(k_0+i)|_{\hat{P}_1} \leq \lambda_1 |\Delta x(k_0)|_{\hat{P}_1} + \beta_1(k_0) \max_{y^{ss}(k) \in Y^{ss}(k)} |r - y^{ss}(k_0)|_{\hat{P}_2} \\ |r - y^{ss}(k_0+i)|_{\hat{P}_2} \leq \lambda_2 |\Delta x(k_0)|_{\hat{P}_1} + \beta_2(k_0) \max_{y^{ss}(k) \in Y^{ss}(k)} |r - y^{ss}(k_0)|_{\hat{P}_2} \end{cases}$$

go to Step 1; otherwise, set $i = i + 1$ and go to Step 2.

We have the following result.

Theorem 27 (State Feedback) *Assume that A is stable for all $A \in \mathcal{A}$ and that the steady-state gain matrix, $C \sum_{i=0}^{\infty} A^i B$, satisfies the following condition.*

$$\max_{(A,B,C) \in (\mathcal{A},\mathcal{B},\mathcal{C})} \left\| \hat{P}_2^{-\frac{1}{2}} \left[I - \left(C \sum_{i=0}^{\infty} A^i B \right) W \right] \hat{P}_2^{\frac{1}{2}} \right\| \triangleq \gamma < 1 \quad \text{for some nonsingular } W \tag{7.27}$$

Let

$$\gamma^* = \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C}) \in (\mathcal{A},\mathcal{B},\mathcal{C})} \left\| \hat{P}_2^{-\frac{1}{2}} \left[I - \left(\tilde{C} \sum_{i=0}^{L-1} \tilde{A}^i \tilde{B} + C \sum_{i=1}^{\infty} A^i \tilde{A}^{L-1} \tilde{B} \right) W \right] \hat{P}_2^{\frac{1}{2}} \right\|$$

$$
\begin{aligned}
\eta^* &= \max_{(A,B)} \left\| \hat{P}_2^{-\frac{1}{2}} A^{L-1} B W \hat{P}_1^{\frac{1}{2}} \right\| \\
\lambda_1^* &= \max_{A} \left\| \hat{P}_1^{-\frac{1}{2}} A^{L} \hat{P}_1^{\frac{1}{2}} \right\| \\
\lambda_2^* &= \max_{(C,A),(\tilde{C},\tilde{A})} \left\| \hat{P}_1^{-\frac{1}{2}} \left(C \sum_{i=1}^{\infty} A^i - \tilde{C} \sum_{i=1}^{\infty} \tilde{A}^i \right) \tilde{A}^L \hat{P}_2^{\frac{1}{2}} \right\| \\
\beta_1(k_0) &= \theta(k_0)\eta \\
\bar{\beta}_2(k_0) &= \frac{\max_{y^{ss}(k_0)\in Y^{ss}(k_0)} |y_0^{ss}(k_0) - y^{ss}(k_0)|_{\hat{P}_2} + [1-\theta(k_0)(1-\gamma^*)]|r - y_0^{ss}(k_0)|_{\hat{P}_2}}{\max_{y^{ss}(k_0)\in Y^{ss}(k_0)} |r - y^{ss}(k_0)|_{\hat{P}_2}} \\
\beta_2(k_0) &= \min\left\{1, \bar{\beta}_2(k_0)\right\} \\
\theta(k_0) &= \begin{cases} 0 & \text{if } \min\limits_{\theta(k_0)\in[0,1]} \alpha\beta_1(k_0) + \beta_2(k_0) \geq \xi, \xi \in (\gamma, 1), \alpha > 0 \\ 1 & \text{otherwise} \end{cases}
\end{aligned}
$$

where $y_0^{ss}(k_0)$ *is the steady-state output for the nominal plant assuming* $\Delta u(k_0+j) = 0 \; \forall \; j \geq 0, \hat{P}_1 = \hat{P}_1' > 0$, *and* $\hat{P}_2 = \hat{P}_2' > 0$. *Then*

1. *the optimization problem (7.26) is feasible for all* $\lambda_1 \geq \lambda_1^*, \lambda_2 \geq \lambda_2^*$, *and* $\eta \geq \eta^*$.

2. *there exists an* L *such that the closed loop system with* Robust Tracking Controller MPCC *is robustly asymptotically stable for all* $r \in \mathcal{Y}$ *and* $\lambda_1 \geq \lambda_1^*, \lambda_2 \geq \lambda_2^*, \eta \geq \eta^*, \alpha > 0$, *and* $\xi \in (\gamma, 1)$ *which satisfy the following relations.*

$$
\begin{aligned}
\lambda_1 + \frac{1}{\alpha}\lambda_2 &< 1 \\
\gamma^* + \alpha\eta &< \xi
\end{aligned}
$$

Proof. For notational simplicity but WLOG, assume $\hat{P}_1 = I$ and $\hat{P}_2 = I$.

1. Clearly all we have to show is that the optimization problem (7.26) is feasible for λ_1^*, λ_2^*, and η^*. It suffices to prove the theorem for $i = 1$ where i is defined as in Step 2 of *Robust Tracking Controller MPCC* : if $i \neq 1$, then $u(k_0+i+j|k_0+i) = u(k_0+i+j|k_0+i-1), j = 0, \cdots, H_c - 1$, is clearly a feasible solution. Let $\Delta u(k_0+j|k_0) = 0 \; \forall \; j \geq 1$. We want to show that $\Delta u(k_0|k_0) = \theta(k_0)W(r - y_0^{ss}(k_0))$ is a feasible solution. $r \in \mathcal{Y}$ implies that $u(k_0|k_0) \in \mathcal{U} \; \forall \; \theta(k_0) \in [0,1]$. That constraint

(7.25) is satisfied can be shown as follows.

$$
\begin{aligned}
\max_{(A,B)} |\Delta x(k_0+L|k_0)| &= \max_{(A,B)} |A^L \Delta x(k_0) + A^{L-1} B \Delta u(k_0|k_0)| \\
&\le \max_{(A,B)} |A^L \Delta x(k_0)| + |A^{L-1} B \Delta u(k_0|k_0)| \\
&\le \max_{A} ||A^L||\ |\Delta x(k_0)| + \max_{(A,B)} |A^{L-1} B \theta(k_0) W (r - y_0^{ss}(k_0))| \\
&\le \lambda_1^* |\Delta x(k_0)| + \theta(k_0) \max_{(A,B)} ||A^{L-1} B W||\ |r - y_0^{ss}(k_0)| \\
&= \lambda_1^* |\Delta x(k_0)| + \theta(k_0) \eta^* |r - y_0^{ss}(k_0)| \\
&\le \lambda_1^* |\Delta x(k_0)| + \theta(k_0) \eta^* \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |r - y^{ss}(k_0)|
\end{aligned}
$$

Next we like to show that $\Delta u(k_0|k_0) = \theta(k_0) W (r - y_0^{ss}(k_0))$ is feasible for constraint (7.24).

$$
\begin{aligned}
& \max_{y^{ss}(k_0+L|k_0)} |r - y^{ss}(k_0+L|k_0)| \\
= \ & \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} \left| r - y(k_0) - \left(\tilde{C} \sum_{i=1}^{L} \tilde{A}^i + C \sum_{i=1}^{\infty} A^i \tilde{A}^L \right) \Delta x(k_0) - M_2 \Delta u(k_0|k_0) \right| \\
= \ & \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} \left| r - \left(y(k_0) + \tilde{C} \sum_{i=1}^{\infty} \tilde{A}^i \right) \Delta x(k_0) - M_1 \Delta x(k_0) - M_2 \Delta u(k_0|k_0) \right| \\
= \ & \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} |r - y^{ss}(k_0) - M_1 \Delta x(k_0) - M_2 \Delta u(k_0|k_0)| \\
\le \ & \max_{(A,C),(\tilde{A},\tilde{C})} |M_1 \Delta x(k_0)| + \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} |r - y^{ss}(k_0) - M_2 \Delta u(k_0|k_0)| \\
\le \ & \lambda_2^* |\Delta x(k_0)| + \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} |y_0^{ss}(k_0) - y^{ss}(k_0) + r - y_0^{ss}(k_0) - M_2 \Delta u(k_0|k_0)| \\
\le \ & \lambda_2^* |\Delta x(k_0)| + \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |y_0^{ss}(k_0) - y^{ss}(k_0)| \\
& \quad + \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} |r - y_0^{ss}(k_0) - M_2 \Delta u(k_0|k_0)| \\
= \ & \lambda_2^* |\Delta x(k_0)| + \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |y_0^{ss}(k_0) - y^{ss}(k_0)| \\
& \quad + \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} |[I - \theta(k_0) M_2 W](r - y_0^{ss}(k_0))| \\
\le \ & \lambda_2^* |\Delta x(k_0)| + \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |y_0^{ss}(k_0) - y^{ss}(k_0)| \\
& \quad + \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})} ||I - \theta(k_0) M_2 W||\ |r - y_0^{ss}(k_0))| \\
\le \ & \lambda_2^* |\Delta x(k_0)| + \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |y_0^{ss}(k_0) - y^{ss}(k_0)| + [1 - \theta(k_0)(1 - \gamma^*)] |r - y_0^{ss}(k_0))| \\
= \ & \lambda_2^* |\Delta x(k_0)| + \bar{\beta}_2(k_0) \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |r - y^{ss}(k_0)|
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &= \left(C \sum_{i=1}^{\infty} A^i - \tilde{C} \sum_{i=1}^{\infty} \tilde{A}^i \right) \tilde{A}^L \\
M_2 &= \left(\tilde{C} \sum_{i=0}^{L-1} \tilde{A}^i + C \sum_{i=1}^{\infty} A^i \tilde{A}^{L-1} \right) \tilde{B}
\end{aligned}
$$

If $\bar{\beta}_2(k_0) \geq 1$, then $\beta_2(k_0) = 1$ and $\theta(k_0) = 0$. We have for $\Delta u(k_0|k_0) = 0$,

$$
\max_{y^{ss}(k_0+L|k_0)} |r - y^{ss}(k_0 + L|k_0)| \leq \lambda_2^* |\Delta x(k_0)| + \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |r - y^{ss}(k_0)|
$$

Thus, constraint (7.24) is feasible.

Since the system is stable, the output constraint is feasible for some sufficiently large $\epsilon(k_0)$. This completes the proof that the optimization problem (7.26) is feasible at each sampling time.

2. Since $\eta^* \to 0, \lambda_1^* \to 0, \lambda_2^* \to 0$, and $\gamma^* \to \gamma$ as $L \to \infty$, a finite L exists such that the set $\{\lambda_1, \lambda_2, \eta, \alpha, \xi : \lambda_1 \geq \lambda_1^*, \lambda_2 \geq \lambda_2^*, \eta \geq \eta^*, \alpha > 0, \xi < 1, \lambda_1 + \frac{1}{\alpha}\lambda_2 < 1$, and $\gamma^* + \alpha\eta < \xi\}$ is nonempty. From (7.25), we obtain

$$
\begin{aligned}
|\Delta x(k_0 + L)| &\leq \max_{(A,B)} \left| A^L \Delta x(k_0) + \mathcal{C}_L \Delta U(k_0|i) \right| \\
&\leq \lambda_1 |\Delta x(k_0)| + \beta_1(k_0) \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |r - y^{ss}(k_0)|
\end{aligned}
$$

Using (7.24) yields

$$
\begin{aligned}
\max_{y^{ss}(k_0+L) \in Y^{ss}(k_0+L)} |r - y^{ss}(k_0 + L)| &\leq \max_{y^{ss}(k_0+L|k_0) \in Y^{ss}(k_0+L|k_0)} |r - y^{ss}(k_0 + L|k_0)| \\
&\leq \lambda_2 |\Delta x(k_0)| + \beta_2(k_0) \max_{y^{ss}(k_0) \in Y^{ss}(k_0)} |r - y^{ss}(k_0)|
\end{aligned}
$$

From these expressions, we have

$$
\alpha |\Delta x(k_0 + L)| + \max_{y^{ss}(k_0+L)} |r - y^{ss}(k_0 + L)|
$$

$$\leq [\lambda_1 + \frac{1}{\alpha}\lambda_2]\alpha|\Delta x(k_0)| + [\beta_2(k_0) + \alpha\beta_1(k_0)] \max_{y^{ss}(k_0)} |r - y^{ss}(k_0)|$$
$$\leq \max\{\lambda_1 + \frac{1}{\alpha}\lambda_2, \beta_2(k_0) + \alpha\beta_1(k_0)\} \left[\alpha|\Delta x(k_0)| + \max_{y^{ss}(k_0)} |r - y^{ss}(k_0)|\right]$$

Since $\lambda_2 + \frac{1}{\alpha}\lambda_1 < 1$ and $\beta_2(k_0) + \alpha\beta_1(k_0) \leq 1 \ \forall \ k_0$, $\alpha|\Delta x(k_0)| + \max_{y^{ss}(k_0)} |r - y^{ss}(k_0)|$ is a non-increasing function of k_0 bounded below by zero.[5] In fact, we want to show that $\alpha|\Delta x(k_0)| + \max_{y^{ss}(k_0)} |r - y^{ss}(k_0)|$ must approach zero asymptotically. Suppose $\alpha|\Delta x(k_0)| + \max_{y^{ss}(k_0)} |r - y^{ss}(k_0)| \to c$ where c is some positive constant. This implies, for sufficiently large k_0, $\beta_2(k_0) + \alpha\beta_1(k_0) \to 1$ and from definition $\theta(k_0) = 0$. Thus, $\Delta x(k_0) \to 0$ asymptotically which yields $\max_{y^{ss}(k_0)} |r - y^{ss}(k_0)| - \min_{y^{ss}(k_0)} |r - y^{ss}(k_0)| \to 0$ asymptotically. Simple calculations show that $\beta_2(k_0) + \alpha\beta_1(k_0)|_{\theta(k_0)=1} \to \gamma^* + \alpha\eta$. This together with $\gamma^* + \alpha\eta < \xi < 1$ yields $\theta(k_0) = 1$ which is a contradiction. Thus, $\alpha|\Delta x(k_0)| + \max_{y^{ss}(k_0)} |r - y^{ss}(k_0)|$ must approach zero asymptotically which in turn yields $y(k) \to r$ asymptotically. That the input is also bounded since the objective function is bounded and only m control moves are allowed, $\square$

Remark 44 *Although Theorem 27 was proven when the 2-norm is used for constraints (7.24) and (7.25), Theorem 27 also holds when any other norm is used. However, the expressions in Theorem 27 may have to be modified accordingly.*

Remark 45 *The sufficient condition (7.27) involves steady-state gain ($C\sum_{i=0}^{\infty} A^i B$) only. For SISO systems, the sufficient condition (7.27) becomes that all plants in the set must have the same steady-state gain sign. This condition is also necessary for the existence of stabilizing controllers with integral control ([66]). For MIMO systems with the set of steady-state gains given by*

$$\mathcal{G}^{ss} = \{g^{ss} : g^{ss} = (I + \Delta)g_0^{ss}, |\Delta| \in \mathbf{\Delta}, g_0^{ss} \text{ nonsingular}\} \tag{7.28}$$

where $\mathbf{\Delta}$ is some uncertainty description, then by setting $W^{-1} = g_0^{ss}$ the sufficient

[5]Here k_0 denotes discrete times defined in Step 1 of *Robust Tracking Controller MPCC* .

condition (7.27) becomes

$$\max_{\Delta \in \boldsymbol{\Delta}} \left\| P^{-1} \Delta P \right\| < 1 \qquad \text{for some } P$$

When $\lambda_2^* = 0$, we have the following corollary.

Corollary 9 *Suppose $\lambda_2 = \lambda_2^* = 0$. Suppose the optimization problem (7.26) is solved with only one constraint (7.24) instead of two constraints (7.24) and (7.25). Under the conditions in Theorem 27, the closed loop system with* Robust Tracking Controller MPCC *is robustly asymptotically stable for all $r \in \mathcal{Y}$.*

Proof. Using the constraint (7.24) and following similar arguments as in the proof of Theorem 27, we can show that the output approaches the setpoint asymptotically and the input is bounded. □

Remark 46 *For systems represented by FIR models of order N, choosing $L = N$ results in $\lambda_2^* = 0$.*

We have the following corollary for all constant disturbances whose steady-state gains are such that offset-free tracking is possible.

Corollary 10 *Theorem 27 holds as well for all constant disturbances whose steady-state gains are such that offset-free tracking is possible.*

7.6 Computation of Control Moves

In this section, we consider systems represented by FIR models. We show that, for a broad class of uncertainty descriptions, the optimization problems (7.18) and (7.26) can be cast as quadratic programs of moderate size.

7.6.1 FIR Models

It is well known (see, for example, Lee et al. [57]) that FIR models can be represented as state-space models. However, their development would result in an A matrix with an integrator and the results in the previous sections would not apply. An alternative state-space representation of an FIR model that would result in a stable A matrix is introduced here.

Consider an FIR model with N impulse response coefficients and denote the i^{th} impulse response coefficient by g_i. For notational simplicity, let us assume that the system is square with $n_y = n_u$ inputs and outputs, i.e. $g_i \in \Re^{n_u \times n_u}$ $\forall\ i = 1, \cdots, N$.

Let

$$x(k) = [u^T(k-N+1)\ \cdots\ u^T(k-1)\ y^T(k)]^T$$

Then

$$\begin{aligned}
A &= \left[\begin{array}{ccccc|c} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline g_N & g_{N-1} & g_{N-2} & \cdots & g_2 & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & 0 \end{array}\right] \in \Re^{[(N-1)n_u+n_y]\times[(N-1)n_u+n_y]} \\
B &= [0\ \cdots\ 0\ I\ g_1]' \in \Re^{[(N-1)n_u+n_y]\times n_u} \\
C &= [0\ \cdots\ 0\ I] \in \Re^{n_u \times [(N-1)n_u+n_y]}
\end{aligned}$$

From these expressions, we have

$$A^i = \begin{bmatrix} A_{11}^i & 0 \\ A_{21}A_{11}^{i-1} & 0 \end{bmatrix}$$

where

$$A_{11}^{i} = \begin{bmatrix} \overbrace{0 \cdots 0}^{i} & I & \cdots & 0 \\ & & \ddots & \\ 0 \cdots 0 & 0 & \cdots & I \\ 0 \cdots 0 & 0 & \cdots & 0 \\ & \vdots & & \\ 0 \cdots 0 & 0 & \cdots & 0 \end{bmatrix} \in \Re^{(N-1)n_u \times (N-1)n_u}$$

$$A_{21}A_{11}^{i-1} = \begin{bmatrix} \overbrace{0 \cdots 0}^{i-1} & g_N & \cdots & g_{i+1} \end{bmatrix} \Re^{n_y \times (N-1)n_u}$$

which yields

$$\mathcal{C}_L = \begin{bmatrix} A^{L-1}B & A^{L-2}B & \cdots & B \end{bmatrix} = \begin{bmatrix} & 0 & \\ & I & \\ g_L & \cdots & g_1 \end{bmatrix} \in \Re^{[(N-1)n_u+n_y] \times nn_u} \tag{7.29}$$

7.6.2 Uncertainty Descriptions

Although the results presented in the previous sections hold for any uncertainty descriptions, the complexity of the optimization problems depends on uncertainty descriptions. Here we introduce a class of uncertainty descriptions for which the optimization problems can be cast as quadratic programs of moderate size. Consider the following set of impulse response coefficients.

$$\mathcal{G} = \left\{ g : g = \bar{g} + \sum_{j=1}^{l} \Delta_j V_j, \bar{g} \in \Re^{n_y \times n_u N}, V_j \in \Re^{n_y \times n_u N}, \Delta_j \in \mathbf{\Delta},\ j = 1, \cdots, l \right\} \tag{7.30}$$

where

$$
\begin{aligned}
\mathbf{\Delta} &= \left\{\Delta : \Delta = \operatorname{diag}\{\delta_1, \cdots, \delta_{n_y}\}, \delta_i \in \Re \text{ and } |\delta_i| \leq 1, i = 1, \cdots, n_y\right\} \\
\bar{g} &= [\bar{g}_N \ \cdots \ \bar{g}_1] \\
g &= [g_N \ \cdots \ g_1] \\
V_j &= [V_{jN} \ \cdots \ V_{j1}] \ \ \forall\, j = 1, \cdots, l, V_{ji} \in \Re^{n_y \times n_u} \ \forall\, i, j
\end{aligned}
$$

We can prove the following results for this set.

Lemma 13 *The constraint*

$$\max_{g \in \mathcal{G}} |a + bz + gz|_1 \leq c \tag{7.31}$$

where a, b, and c are constants of appropriate dimensions, is equivalent to a set of linear constraints.

Proof. By the special structure of $\mathbf{\Delta}$, we have

$$
\begin{aligned}
\max_{g \in \mathcal{G}} |a + bz + gz|_1 &= \max_{\Delta_i \in \mathbf{\Delta}, i=1,\cdots,l} \left| a + bz + \left(\bar{g} + \sum_{i=1}^{l} \Delta_i V_i\right) z \right|_1 \\
&= \max_{\Delta_i \in \mathbf{\Delta}, i=1,\cdots,l} \left| a + bz + \bar{g}z + \sum_{i=1}^{l} \Delta_i V_i z \right|_1 \\
&= |a + bz + \bar{g}z|_1 + \sum_{i=1}^{l} |V_i z|_1
\end{aligned}
$$

which yields

$$\max_{g \in \mathcal{G}} |a + bz + gz|_1 \leq c \Leftrightarrow |a + bz + \bar{g}z|_1 + \sum_{i=1}^{l} |V_i z|_1 \leq c$$

Let $\xi_i = \text{abs}(V_i z), i = 1, \cdots, l$, and $\eta = \text{abs}(a + bz + \bar{g}z)$ where $\text{abs}(z)$ denotes the absolute value of the vector z. Then $\max_{g \in \mathcal{G}} |a + gz|_1 \leq b$ is feasible if and only if

$$\begin{cases} \sum_{i=1}^{l} \sum_{j=1}^{n_y} \xi_{ij} + \sum_{i=1}^{n_y} \eta_i & \leq & c \\ -\xi_i & \leq & V_i z, \quad i = 1, \cdots, l \\ V_i z & \leq & \xi_i, \quad i = 1, \cdots, l \\ -\eta & \leq & a + bz + \bar{g}z \\ a + bz + \bar{g}z & \leq & \eta \end{cases}$$

is feasible. The latter is clearly a set of linear constraints. □

Remark 47 *The total number of linear constraints representing (7.31) is* $2n_y(l + 1) + 1$ *which is moderate.*

Lemma 14

$$\sum_{i=1}^{n} \delta_i z_i \leq b \ \forall \ \delta_i \in \Re, |\delta_i| \leq 1 \ \forall \ i \Leftrightarrow \sum_{i=1}^{n} \text{abs}(z_i) \leq b$$

where z_i *and* b *are vectors of the same dimension.*

Proof. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Let $\xi = \sum_{i=1}^{n} \text{abs}(z_i)$. Assume $\sum_{i=1}^{n} \delta_i z_i \leq b \ \forall \ |\delta_i| \leq 1$. Suppose the first element of ξ is larger than the first element of b, i.e. $\xi_1 > b_1$. Then there exists $|\delta_i| \leq 1 \ \forall \ i$ such that $\sum_{i=1}^{n} \delta_i z_{i1} > b$ which is a contradiction. □

Lemma 15 *The constraint*

$$Hz + Fgz \leq f \ \forall \ g \in \mathcal{G} \tag{7.32}$$

where $H, F,$ *and* f *are constants of appropriate dimensions can be represented as a set of linear inequalities.*

Proof.

$$\begin{aligned} Fgz &= F(\bar{g} + \sum_{j=1}^{l} \Delta_j V_j)z \\ &= F\bar{g}z + \sum_{j=1}^{l} F\Delta_j V_j z \\ &= F\bar{g}z + \sum_{j=1}^{l} [\delta_{j1}F_1 \ \cdots \ \delta_{jn_u}F_{n_u}]V_j z \\ &= F\bar{g}z + \sum_{j=1}^{l}\sum_{i=1}^{n_u} \delta_{ji}Fi(V_j z)_i \end{aligned}$$

where $(V_j z)_i$ denotes the i^{th} row vector of $V_j z$. Using Lemma 14, we have

$$Hz + Fgz \leq f \ \forall \ g \in \mathcal{G} \Leftrightarrow (H + F\bar{g})z + \sum_{i=1}^{ln_u} \text{abs}(v_i) \leq b$$

where v_i is defined appropriately. By defining $\xi_i = \text{abs}(v_i)$ and following similar steps as in the proof of Lemma 13, we can show that $Hz + Fgz \leq f \ \forall \ g \in \mathcal{G}$ can be represented as a set of linear inequalities. □

7.6.3 Casting Optimization Problems as QPs

We make the following assumptions.

Assumption 1 Systems are represented by FIR models.

Assumption 2 Uncertainty is described by (7.30) and $g \in \mathcal{G}$.

Assumption 3 $\hat{P}, \hat{P}_1$, and $\hat{P}_2$ are diagonal.

Assumption 4 1−norm is used for constraints (7.19), (7.24), and (7.25).

The following theorem states that the output constraint can be represented by a set of linear inequalities.

Theorem 28 *The output constraint $x(k+i|k) \in \mathcal{X}_{\epsilon(k)} \ \forall \ i \geq 1$ can be cast as a set of linear inequalities.*

Proof. Since the system is represented as an FIR model, with H_c control moves the system settles down in exactly $N + H_c$ steps, i.e. after which the output does not change. The output constraints over the infinite horizon can be replaced with the output constraints over a finite horizon of length $N + H_c$, i.e.

$$x(k+i|k) \in \mathcal{X}_{\epsilon(k)} \ \forall \ i \geq 1 \Longleftrightarrow x(k+i|k) \in \mathcal{X}_{\epsilon(k)}, i = 1, \cdots, N + H_c$$

Simple calculations yield

$$x(k+i|k) = \begin{bmatrix} \overbrace{0 \cdots 0}^{i} & I & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 \cdots 0 & 0 & \cdots & I & 0 \\ 0 \cdots 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & & \\ 0 \cdots 0 & 0 & \cdots & 0 & 0 \\ 0 \cdots g_N & g_{N-1} & \cdots & g_{i+1} & 0 \end{bmatrix} x(k_0) + \begin{bmatrix} & 0 & \\ & I & \\ g_i & \cdots & g_1 \end{bmatrix} \begin{bmatrix} u(k) \\ \vdots \\ u(k+i-1) \end{bmatrix}$$

After some algebra, we can put the constraint

$$[F_x \ \ F_u] \begin{bmatrix} x(k+i|k) \\ u(k+i|k) \end{bmatrix} \leq f + \epsilon(k) \ \forall \ \Delta_j \in \mathbf{\Delta}, j = 1, \cdots, l$$

into the form (7.32). Then direct application of Lemma 15 yields the desired result. $\square$

The following theorem states that the robust stability constraint (7.19) can be represented by a set of linear inequalities.

Theorem 29 *Under Assumptions* $1 - 4$*, the robust stability constraint (7.19) can be represented as a set of linear constraints.*

Proof. With Assumptions $1-4$, after some algebra, the robust stability constraint (7.19) becomes

$$\max_{g \in \mathcal{G}} \left| \hat{P} \left[\begin{array}{cccccc} \overbrace{0 \cdots 0}^{L} & I & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 \cdots 0 & 0 & \cdots & I & 0 \\ 0 \cdots 0 & 0 & \cdots & 0 & 0 \\ & \vdots & & & \\ 0 \cdots 0 & 0 & \cdots & 0 & 0 \\ 0 \cdots g_N & g_{N-1} & \cdots & g_{L+1} & 0 \end{array} \right] x(k_0) + \right. \tag{7.33}$$

$$\left. \left[\begin{array}{c} 0 \\ I \\ g_L \cdots g_1 \end{array} \right] \left[\begin{array}{c} u(k_0) \\ \vdots \\ u(k_0+i-2) \\ u(k_0+i-1|k_0+i-1) \\ \vdots \\ u(k_0+L-1|k_0+L-1) \end{array} \right] \right|_1 \leq \lambda |\hat{P}x(k_0)|_1$$

The right-hand-side is known at each sampling time. Since diagonal $\hat{P}$ commutes with Δ, it is WLOG to assume $\hat{P} = I$. (7.33) can clearly be put into the form (7.31). Then direct application of Lemma 13 gives the desired result. □

Remark 48 *Since $A^N = 0 \ \forall \ A \in \mathcal{A}$, $\lambda^*(N) = 0$. With $L = N$ the optimization problem (7.18) is feasible for all $\lambda \in [0, 1)$.*

The optimization problem (7.18) can be solved as a quadratic program since the objective function is quadratic and the constraints are linear. That constraints (7.24) and (7.25) can be cast as sets of linear constraints is stated as follows.

Theorem 30 *Under Assumptions* $1-4$*, constraints (7.24) and (7.25) can be represented as a set of linear constraints.*

Proof. WLOG, assume $\hat{P}_1 = I$ and $\hat{P}_2 = I$. For systems represented by FIR models, we have

$$\begin{aligned} M_1 = \tilde{C}\sum_{i=1}^{L}\tilde{A}^i + C\sum_{i=1}^{\infty}A^i\tilde{A}^L &= \tilde{D} + D \\ M_2 = \left[\tilde{C}\sum_{i=0}^{L-1}\tilde{A}^i + C\sum_{i=1}^{\infty}A^i\tilde{A}^{L-1} \quad \cdots \quad \tilde{C} + C\sum_{i=1}^{\infty}A^i\right]\tilde{B} &= \tilde{E} + E \end{aligned}$$

where

$$\begin{aligned} \tilde{D} &= \begin{bmatrix} \tilde{g}_N & \sum_{i=N-1}^{N}\tilde{g}_i & \cdots & \sum_{i=N-L+1}^{N}\tilde{g}_i & \sum_{i=N-L}^{N-1}\tilde{g}_i & \cdots & \sum_{i=2}^{L+1}\tilde{g}_i & 0 \end{bmatrix} \\ D &= \begin{bmatrix} 0 & 0 & \cdots & 0 & g_N & \cdots & \sum_{i=L+2}^{N} g_i & 0 \end{bmatrix} \\ \tilde{E} &= \begin{bmatrix} \sum_{i=1}^{L}\tilde{g}_i & \sum_{i=1}^{L-1}\tilde{g}_i & \cdots & \sum_{i=1}^{2}\tilde{g}_i & \tilde{g}_1 \end{bmatrix} \\ E &= \begin{bmatrix} \sum_{i=L+1}^{N} g_i & \sum_{i=L}^{N} g_i & \cdots & \sum_{i=3}^{N} g_i & \sum_{i=2}^{N} g_i \end{bmatrix} \end{aligned}$$

By the uncertainty description (7.30), we have

$$\sum_{j=n_1}^{n_2} g_j = \sum_{j=n_1}^{n_2}\left(\bar{g}_j + \sum_{i=1}^{l}\Delta_i V_{ij}\right) = \sum_{j=n_1}^{n_2}\bar{g}_j + \sum_{j=n_1}^{n_2}\sum_{i=1}^{l}\Delta_i V_{ij} = \sum_{j=n_1}^{n_2}\bar{g}_j + \sum_{i=1}^{l}\Delta_i\sum_{j=n_1}^{n_2}V_{ij}$$

which is of the same form as that of (7.30). $\max_{y^{ss}(k_0)}|r - y^{ss}(k_0)|_1$ can be determined as follows.

$$\begin{aligned} \max_{y^{ss}(k_0)}|r - y^{ss}(k_0)|_1 &= \max_{g\in\mathcal{G}}\left|r - y(k_0) - \begin{bmatrix} g_N & \sum_{j=N-1}^{N} g_j & \cdots & \sum_{j=2}^{N} g_j & 0 \end{bmatrix}\Delta x(k_0)\right|_1 \\ &= \left|r - y(k_0) - \begin{bmatrix} \bar{g}_N & \sum_{j=N-1}^{N}\bar{g}_j & \cdots & \sum_{j=2}^{N}\bar{g}_j & 0 \end{bmatrix}\Delta x(k_0)\right|_1 \\ &\quad + \sum_{i=1}^{l}\left|\begin{bmatrix} V_{iN} & \sum_{j=N-1}^{N} V_{ij} & \cdots & \sum_{j=2}^{N} V_{ij} & 0 \end{bmatrix}\Delta x(k_0)\right|_1 \end{aligned}$$

The constraint (7.24) can be cast as a set of linear constraints as shown in Theorem 29. We can write the left-hand-side of the constraint (7.25) as follows.

$$\max_{y^{ss}(k_0+L|k_0)\in Y^{ss}(k_0+L|k_0)} |r - y^{ss}(k_0 + L|k_0)|_1$$

$$\begin{aligned}
&= \max_{(A,B,C),(\tilde{A},\tilde{B},\tilde{C})\in(\mathcal{A},\mathcal{B},\mathcal{C})} |r - y(k_0) - M_1\Delta x(k_0) - M_2\Delta U(k_0|i)|_1 \\
&= \max_{g,\tilde{g}\in\mathcal{G}} \left|r - y(k_0) - (\tilde{D} + D)\Delta x(k_0) - (\tilde{E} + E)\Delta U(k_0|i)\right|_1
\end{aligned}$$

which can be clearly put into the form (7.31). Direct application of Lemma 13 results in that the constraint (7.25) can be represented by a set of linear constraints. □

With $\hat{\Phi}_k(A_0, B_0, C_0)$ quadratic in control moves and linear constraints, we can solve the optimization problem (7.26) as a quadratic program.

7.7 Examples

Three examples are presented here to demonstrate the characteristics of the proposed method. In Example 8, we consider the problem of robust stabilization to the origin. The set of linear time-invariant systems is such that the robust stability constraint is *not* feasible with $L = 1$. Both Examples 9 and 10 deal with output tracking. Example 9 considers the same problem which was used in Chapter 6 and compares the differences between the methods presented in Chapter 6 and in this chapter. The Idle Speed Control problem [94] described in Example 10 has been studied extensively by researchers using various control methods [43]. However, none of these methods deals with the input constraint which is present. Using the results developed here, we can design a controller which guarantees robust stability with the input constraint.

Example 8 The set of plants is given below.

$$\mathcal{A} = \{A : A = \alpha A_1 + (1 - \alpha)A_2, 0 \leq \alpha \leq 1\} \tag{7.34}$$

where

$$A_1 = \begin{bmatrix} 0.7 & 0.6 \\ 0.6 & -0.7 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.4 & 0.4 \\ -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The stability constraint may be infeasible for $L = 1$ and $\lambda \leq 1$. However, for $L = 2$ and $P = \begin{bmatrix} 3.85 & -1.08 \\ -1.08 & 1.67 \end{bmatrix}$, the stability constraint is feasible for all $\lambda \geq 0.742$. The input is constrained between ± 1. The initial condition is $x_0 = [2 \;\; 2]'$. By Theorem 26, global asymptotic stability is guaranteed with the following tuning parameters.

$$H_c = 5, H_p = 10, L = 2, P = \begin{bmatrix} 3.85 & -1.08 \\ -1.08 & 1.67 \end{bmatrix}, \Gamma_x = I, \Gamma_u = 0.1I, \Gamma_{\Delta u} = 0$$

Figure 7.1 shows performance for the nominal plant, *i.e.* $A = \frac{1}{2}(A_1 + A_2)$ while Figure 7.2 shows performance for $A = A_2$.

Remark 49 *One way to design a robustly stabilizing linear controller for a set of linear plants is via the Linear Matrix Inequality (LMI) technique [6]. The basic idea behind the LMI technique is that a linear controller is robustly stable if there exists a positive definite matrix P such that $P - A'PA < 0$ for all $A \in \mathcal{A}$. However, this condition is not necessary. For the set of systems described by (7.34), there does not exist such a positive matrix P satisfying the inequality. This is because the LMI technique assumes the system to be* time-varying. *Thus, designing a robustly stabilizing controller via the LMI technique[49] for a* time-invariant *system may not be possible.*

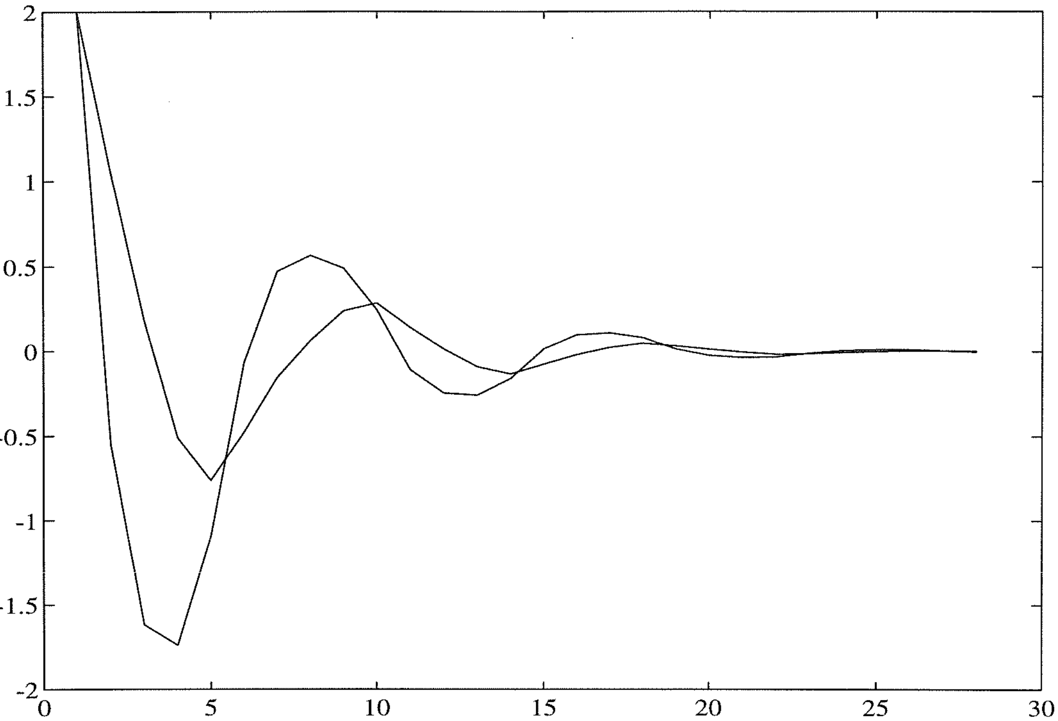

Figure 7.1: Example 8—Nominal responses ($A = \frac{1}{2}(A_1 + A_2)$)

Example 9 The set of models is described by

$$\mathcal{G} = \{G(q) : G(q) = \delta_2 G_0(q) + \delta_1 [G_0(q) - G_1(q)], 0 \leq \delta_1 \leq 0.5 \text{ and } 0.5 \leq \delta_2 \leq 1.5\} \tag{7.35}$$

where $G_0(q) = \frac{0.75}{q-0.75}$ and $G_1 = \frac{0.75(-q+1.8)}{(q-0.75)(q-0.2)}$. Here δ_1 and δ_2 can be interpreted as follows: δ_1 accounts for possible unmodelled dynamics while δ_2 accounts for the gain uncertainty.

$\mathcal{G}$ can be put into the following form:

$$\mathcal{G} = \left\{G(q) : G(q) = \left[\frac{5}{4}G_0(q) - \frac{1}{4}G_1(q)\right] + \bar{\delta}_2 \frac{1}{2} G_0(q) + \bar{\delta}_1 \frac{1}{4}\left[G_0(q) - G_1(q)\right], |\bar{\delta}_i| \leq 1\right\}$$

G_0 and G_1 are truncated by FIR models of order 15. We can put the set of models into the form (7.30) with $l = 2$ and $\mathbf{\Delta} = \{\Delta : \Delta \in \Re \text{ and } |\Delta| \leq 1\}$. Thus the optimization problem can be solved as a quadratic program. For setpoint tracking, by Theorem 27 and Corollary 10, global asymptotic stability is guaranteed with the

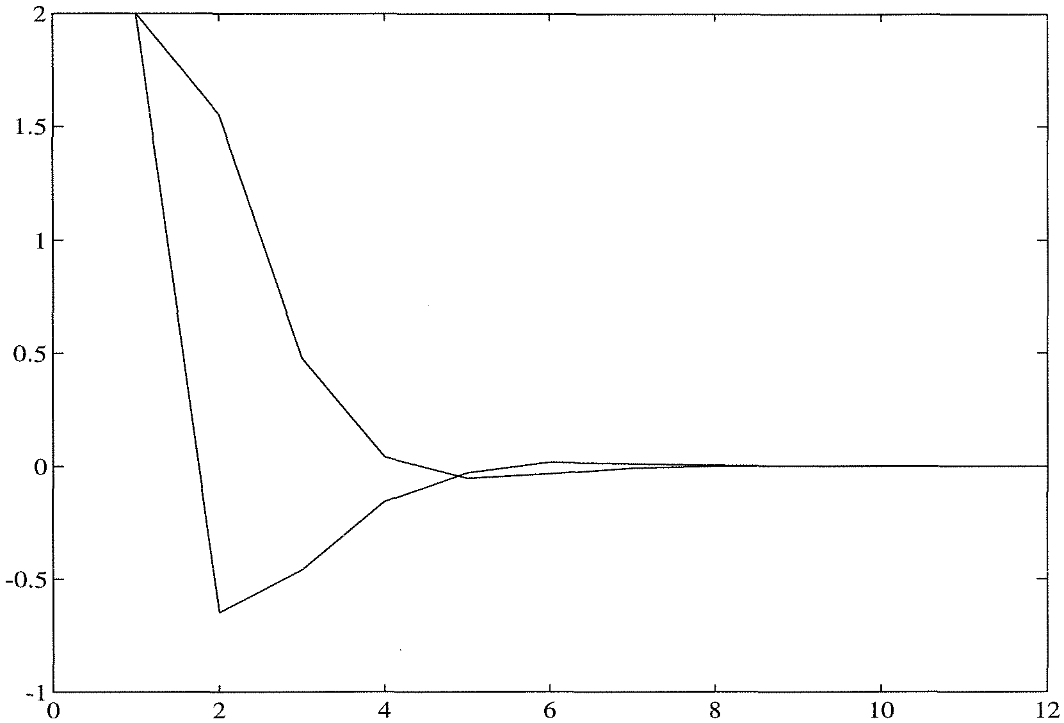

Figure 7.2: Example 8—Responses for $A = A_2$

following tuning parameters.

$$L = 15, H_c = 2, H_p = 4, \Gamma_{\Delta u} = 0.5, \Gamma_y = 1, \Gamma_u = 0, \lambda_2 = 0$$

The input is constrained between ± 0.8. Figure 7.3 shows the output responses for a unit-step setpoint change for the nominal plant ($\bar{\delta}_1 = 0$ and $\bar{\delta}_2 = 0$). Performances for the four extreme plants depicted in Figure 7.4 are worse than that of the nominal plant but the closed loop system is asymptotically stable. This is expected since the objective here is to optimize nominal performance subject to robust stability constraints. Figure 7.5 compares the performance for the four extreme plants obtained by using the method proposed in this chapter (which assumes that the system is LTI) to that obtained by using the method proposed in Chapter 6 (which assumes that the system is LTV). Although the method presented here does *not* attempt to optimize the worst case performance, the performance obtained is at least comparable, if not better, than the method proposed in Chapter 6. It should be pointed out that the method proposed in Chapter 6 applies to LTV systems.

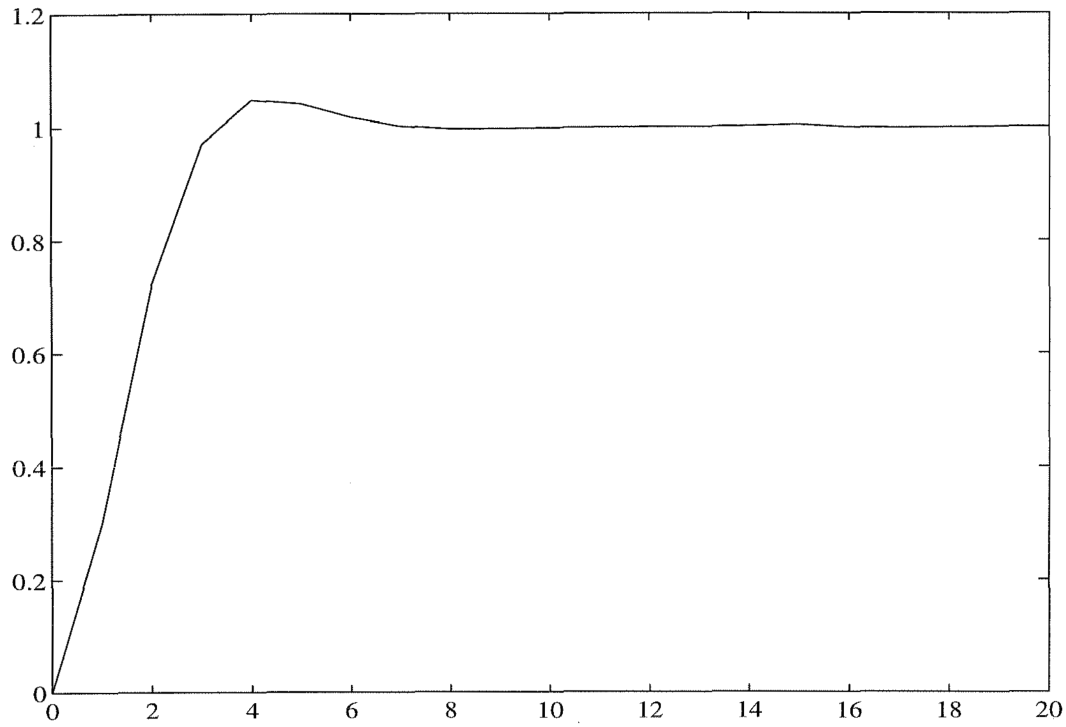

Figure 7.3: Example 9—Nominal responses

Example 10 —*Idle Speed Control [94]*

Consider the system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{21} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_d \\ 0 \end{bmatrix} w \tag{7.36}$$

where y_1 is engine rpm, y_2 and u_2 are spark advance, u_1 is bypass valve, w is torque load (unmeasured disturbance), and G_{11}, G_{21} and G_d are the corresponding transfer functions. After appropriate scaling, the constraints on spark advance become ± 0.7, *i.e.* $|u_2| \leq 0.7$.

Here we consider two different operating conditions (transmission in neutral and drive positions) and the models for the two plants are taken from [43]. Plant #1

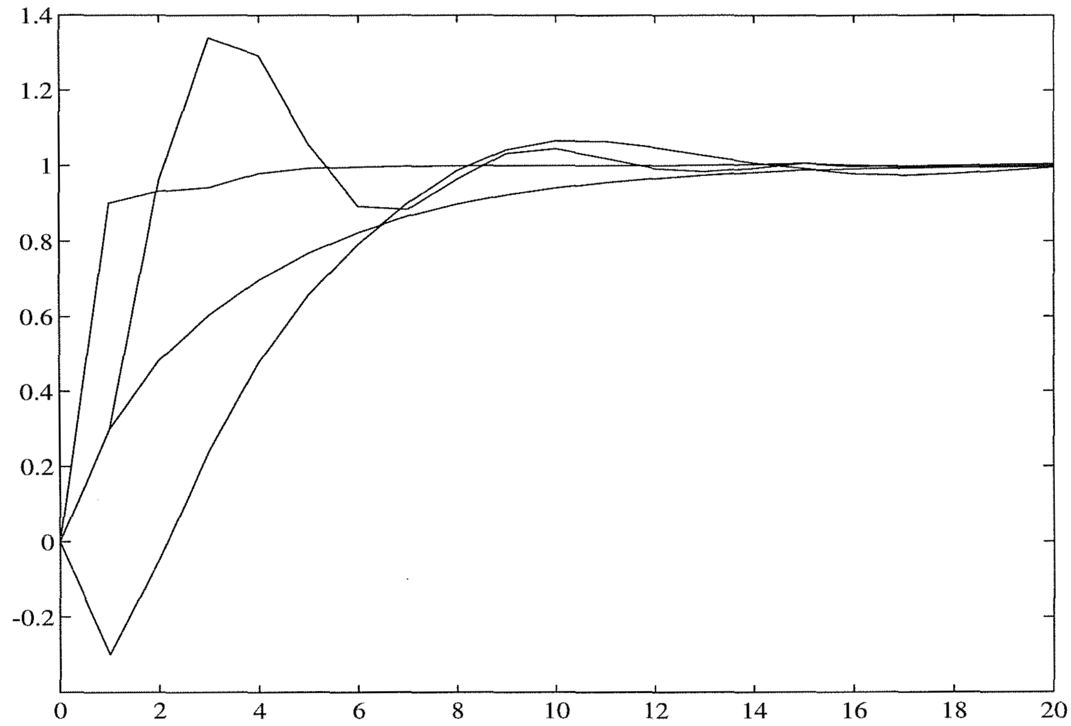

Figure 7.4: Example 9—Responses for other plants

corresponds to operation at 800 rpm and a load of 30 Nm (transmission in drive position) and is given by

$$G_{11} = \frac{9.62e^{-0.16s}}{s^2 + 2.4s + 5.05} \tag{7.37}$$

$$G_{21} = \frac{15.9(s+3)e^{-0.04s}}{s^2 + 2.4s + 5.05} \tag{7.38}$$

Plant #2 corresponds to operation at 800 rpm and zero load (transmission in neutral position) and is given by

$$G_{11} = \frac{20.5e^{-0.16s}}{s^2 + 2.2s + 12.8} \tag{7.39}$$

$$G_{21} = \frac{47.6(s+3.5)e^{-0.04s}}{s^2 + 2.2s + 12.8} \tag{7.40}$$

We first truncate both plants by FIR models with 24 coefficients and sampling time of 0.1. The nominal model ($\bar{G}$) equals $\frac{G_1+G_2}{2}$ and the uncertainty (V) equals

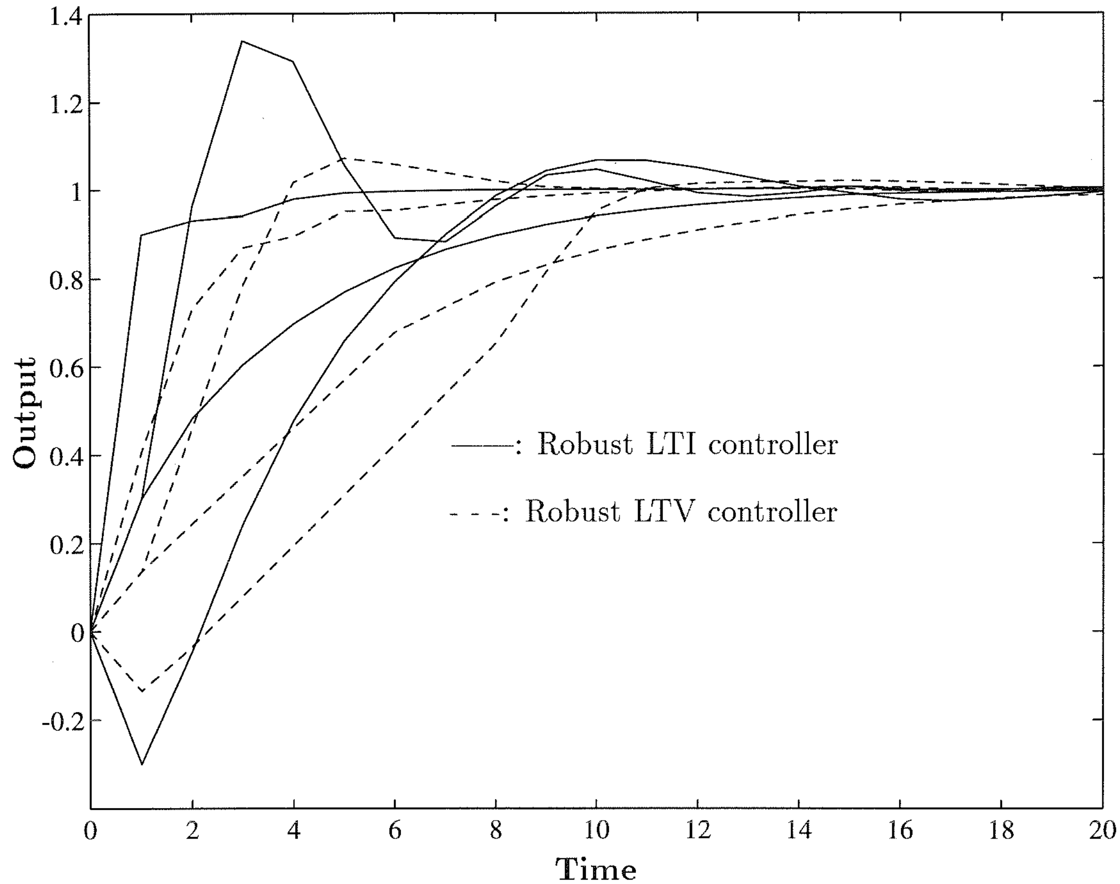

Figure 7.5: Comparison of robust LTI and robust LTV controllers

$\frac{G_1 - G_2}{2}$. Thus, the set of plants can be described as follows.

$$\mathcal{G} = \{G(q) : G(q) = \bar{G}(q) + \delta V(q), |\bar{\delta}| \leq 1\}$$

With these preliminaries, we can cast the optimization problem as a quadratic program. The following tuning parameters are used.

$$L = 24, H_c = 5, H_p = 10, \Gamma_{\Delta u} = 0.5I, \Gamma_y = I, \Gamma_u = 0, \lambda_2 = 0$$

By Theorem 27 and Corollary 10, global asymptotic stability is guaranteed. Figure 7.6 shows nominal performance for a setpoint change while robust performance is shown in Figure 7.7.

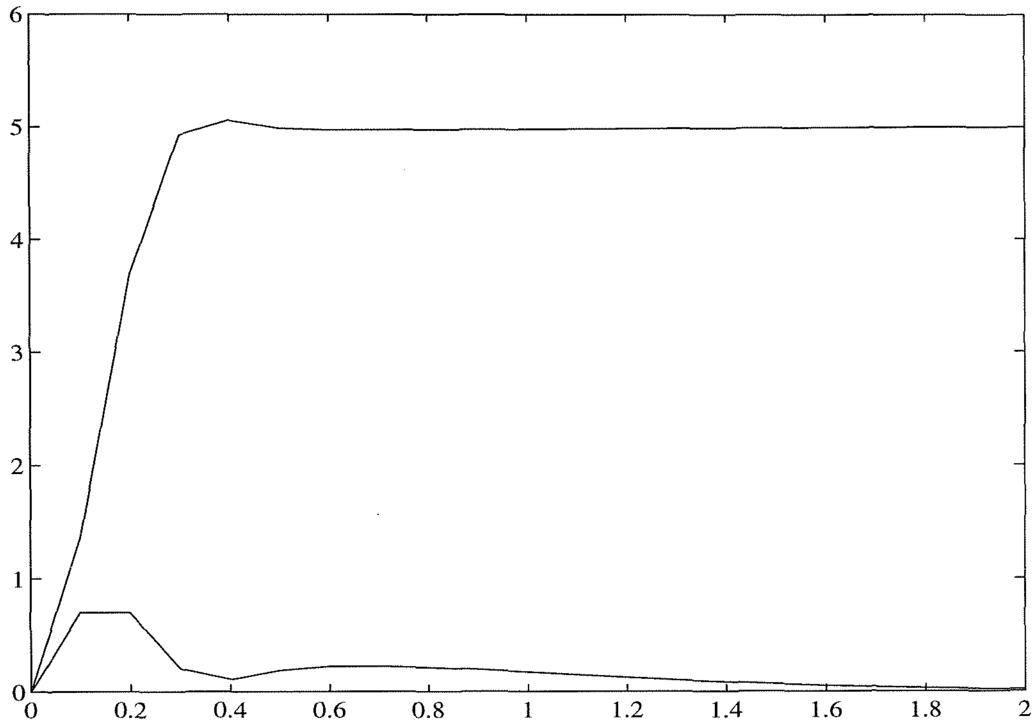

Figure 7.6: Example 10—Nominal Response

7.8 Conclusions

In this chapter, we proposed an MPC algorithm which optimizes performance subject to stability constraints for controlling linear time-invariant discrete-time systems with "hard" input constraints and "soft" output constraints. In the nominal case, we showed that global asymptotic stability is guaranteed for both state feedback and output feedback for linear time-invariant stable systems. Furthermore, global asymptotic stability is preserved for all asymptotically constant disturbances. The algorithm was then generalized to the robust case. We showed that robust global asymptotic stability is guaranteed for a set of linear time-invariant stable systems. When the system is represented by a step response model, we showed that the optimization problem with an appropriate objective function can be cast as a quadratic program for a broad class of uncertainty descriptions.

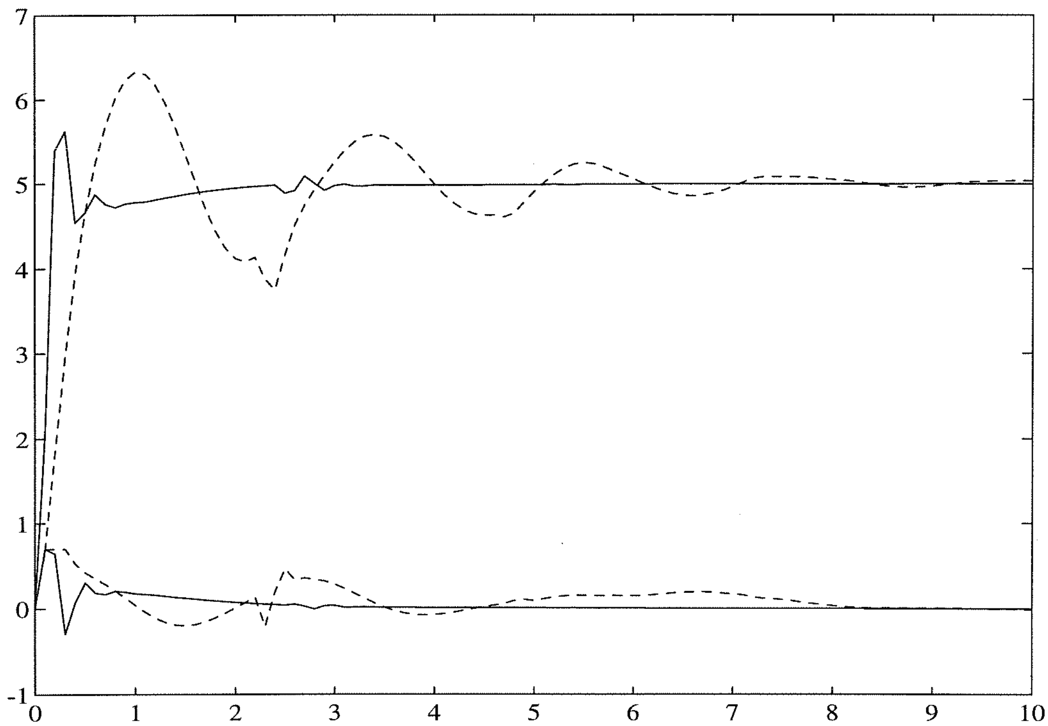

Figure 7.7: Example 10—Response for Other Plants (Solid: Plant # 1; Dashed: Plant # 2

Chapter 8 Summary of Contributions and Suggestions for Future Work

8.1 Summary of Contributions

Although a rich theory has been developed for the robust control of linear systems *without* constraints, *very little* was known for the robust control of linear systems *with* constraints. In this thesis, we have developed the *first* general synthesis techniques for designing controllers for linear discrete-time systems subject to constraints with robust stability and robust performance guarantees.

In Chapters 3, 4, and 5, a complete theory has been developed for linear systems *without* model uncertainty with mixed constraints—hard input constraints and soft output constraints. For stable systems (Chapter 3), we showed that global asymptotic stability with the Infinite Horizon MPC with Mixed Constraints (IHMPCMC) algorithm is guaranteed for both state feedback and output feedback cases. The on-line optimization problem can be cast as a *finite* dimensional program even though the output constraints are specified over an infinite horizon.

The problem of global stabilization of linear systems with poles on the unit circle has attracted much attention recently. Based on the growth rate of the set of states reachable with unit-energy inputs, we showed in Chapter 4 that a discrete-time controllable linear system is globally controllable to the origin with *energy-bounded* inputs (i.e. $\sum_{i=1}^{\infty} u(i)^T u(i) < \infty$) if and only if all its eigenvalues lie in the closed unit disk. These results imply that the IHMPCMC algorithm is semi-globally stabilizing for a sufficiently long input horizon if and only if the controlled system is stabilizable and all its eigenvalues lie in the closed unit disk. The disadvantage of this IHMPCMC algorithm is that the input horizon necessary for stabilization depends on the initial condition and can be arbitrarily large. As a result, we proposed an implementable

IHMPCMC algorithm. We showed that with this algorithm a discrete-time linear system with n poles on the unit disk (with any multiplicity) can be globally stabilized if the input horizon is larger than n. For pure integrator systems, this condition is also necessary. Moreover, we showed that global asymptotic stability is preserved for any asymptotically constant disturbance entering at the plant input.

Global stabilization of unstable linear system with constraints is not possible. It is important to characterize the domain of attraction, i.e. the set of all initial conditions for which stabilization is possible, for such systems. However, very little work has been done. In Chapter 5, we analyzed and characterized the domain of attraction for a linear unstable discrete-time system with bounded controls. An algorithm was proposed to construct the domain of attraction. We showed that the IHMPCMC algorithm generates a class of (nonlinear) control laws that stabilize the system for all initial conditions in the domain of attraction.

In Chapter 6, we generalized the robust MPC algorithm proposed by Campo and Morari [10] for control of linear time-varying systems (represented by FIR models) with constraints. We showed that with this scheme robust Bounded-Input Bounded-Output stability is guaranteed. Both necessary and sufficient conditions for global asymptotic robust stability are stated. Furthermore, we showed that robust global asymptotic stability is preserved for a class of asymptotically constant disturbances entering at the plant output. Although these results hold for any uncertainty description expressed in the time-domain, there is a trade-off between the generality of the uncertainty description and the computational complexity of the resulting optimization problem. For a broad class of uncertainty descriptions, we showed that the optimization problem can be cast as a linear program of moderate size.

In Chapter 7, we considered linear time-invariant systems. We proposed a *novel* MPC algorithm which optimizes performance subject to stability constraints for linear systems with mixed constraints (i.e. "hard" input constraints and "soft" output constraints). In the nominal case, we showed that global asymptotic stability is guaranteed for both state feedback and output feedback for linear time-invariant stable systems. Furthermore, global asymptotic stability is preserved for all asymptotically constant disturbances. The algorithm was then generalized to the robust case. We showed that robust global asymptotic stability is guaranteed for a set of linear time-invariant stable systems. When the system is represented by a Finite Impulse Response model, we showed that the optimization problem can be cast as a quadratic program of moderate size for a broad class of uncertainty descriptions. The theory was successfully applied to the Idle Speed Control problem.

Most of the AWBT schemes appeared in the literature over the years have been developed for SISO systems. Features unique to MIMO systems such as gain directionality make these methods fail. In Appendix A, a general anti-windup design which optimizes the error between the constrained output and the unconstrained output of the system, applicable to MIMO systems, is developed.

8.2 Suggestions for Future Work

Even though there is little doubt that MPC is the most widely used advanced control technique in process industry, it constitutes only a small portion of all controllers used in process industry. Furthermore, applications in other disciplines are very rare. Work in the following areas is needed for MPC to have widespread acceptance and application in all disciplines.

Modeling The routine application of robust MPC techniques faces many obstacles. One key difficulty which stands in the way of these new techniques, as well as many other advanced control techniques, is the need for a model and the associated uncertainty description to describe the dynamic behavior of the process to be controlled. While many aspects of modeling has been studied for decades, the *modeling needs for control* purposes are largely not understood. Modeling is expensive and time-consuming. It is of key interest to minimize the modeling effort required for a specific control implementation. A clear understanding of the trade-offs between model accuracy and control quality is essential for determining if increased modeling effort is justified.

Computational Complexity Because of the high computational requirements, MPC is typically implemented in a supervisory mode, i.e. on top of the regulatory control systems, on systems with large sampling times, and on systems with a moderate number of inputs and outputs. This explains why MPC is widely used in process industry but *not* in other disciplines. Reducing the computational requirements for MPC will expand MPC's applications to include other challenging processes from other disciplines.

Nonlinear Systems The extension of the basic MPC concept is straightforward and much research has been done on nonlinear MPC. One key difficulty in applying these techniques to a practical control problem is that the required computation is forbiddingly high. Part of the difficulty lies in that most of the techniques are intended for general *nonlinear* systems. For general nonlinear systems *without* constraints, most of control techniques available [24] in the literature require such amount of computation that even designing controllers *off-line* is forbidding for a reasonably nontrivial system. It is important to restrict the class of nonlinear systems such that the computation is manageable and yet the class is rich enough to cover (or approximate reasonably well) a real process.

Process Diagnostics and Monitoring Sensor and actuator faults or failures in sensors and actuators are common in process control applications. Since they can be expressed as additional constraints, they can be handled trivially by MPC *provided* they are recognized. Developing process diagnostic and monitoring tools and incorporating them within the MPC framework will be crucial for maintaining system performance.

Process Applications The synthesis technique developed in Chapter 7 was applied to the Idle Speed Control problem and the results are promising. However, the ultimate effectiveness of any control approach must be judged on the basis of its application to real systems. Applications to real systems will suggest how to modify the theory to improve its applicability.

Appendix A Anti-Windup Design for Internal Model Control

Summary

This appendix considers linear control design for systems with input magnitude saturation. A general anti-windup scheme which optimizes nonlinear performance, applicable to multi-input multi-output systems, is developed. Several examples, including an ill-conditioned plant, show that the scheme provides graceful degradation of performance. The attractive features of this scheme are its simplicity and effectiveness.

A.1 Introduction

Of special interest and common occurrence are systems having control input saturation nonlinearities but which are otherwise linear. Windup problems were originally encountered when using PI/PID controllers for controlling such systems. However, it was recognized later that integrator windup is only a special case of a more general problem. As pointed out by Doyle et al. [26], any controller with relatively slow or unstable modes will experience windup problems if there are actuator constraints. Windup is then interpreted as an inconsistency between the plant input and the states of the controller when the control signal saturates.

The "conditioning technique" as an anti-windup technique was originally formulated by Hanus *et al.* [39, 40] as an extension of the back calculation method of Fertik and Ross [28] to a general class of controllers. Åström and Wittenmark [2] and Åström and Rundqwist [1] proposed that an observer be introduced into the system to estimate the states of the controller in the face of constraints and hence restore consistency between the saturated control signal and the controller states. This observer-based approach represented a significant generalization of the existing anti-windup schemes. Walgama and Sternby [93] have clearly exposed this inherent observer property in a large number of anti-windup schemes. Campo and Morari [11] have derived the Hanus conditioned controller independently as a special case of the observer-based approach.

All these anti-windup schemes have been developed only for single-input single-output (SISO) systems. The extension to multi-input multi-output (MIMO) systems has not been attempted in its entirety. As pointed by Doyle et al. [26], for MIMO controllers, the saturation may cause a change in the plant input direction resulting in disastrous consequences. Through an example, Doyle et al. showed that all of the existing anti-windup schemes failed to work on MIMO systems. It is one of the objectives of this chapter to develop an anti-windup scheme which is applicable to MIMO systems.

The Internal Model Control (IMC) structure [69] (see Figure A.1) was never intended to be an anti-windup scheme. Although stability of P and Q would guarantee global stability, provided that there is no plant-model mismatch, the performance suffers when there are actuator constraints. This is because the controller (Q) is entirely unaware of the effect of its action. In particular, it does not know if and when the manipulated variable (u) saturates. This effect is most pronounced when the IMC controller has fast dynamics which are chopped off by the saturation. Unless the IMC controller is designed to optimize nonlinear performance, it will not give satisfactory performance for the saturating system. The focus of this chapter is to identify this nonlinear performance.

Assumptions and Notations We will assume that the plant is a linear time invariant and stable square system with n inputs and n outputs. For simplicity, we will use the same symbol to denote both the transfer function and the corresponding impulse response model. The meaning should be clear from context. P, $\tilde{P}$, and Q denote the plant, the model of the plant, and the IMC controller, respec-

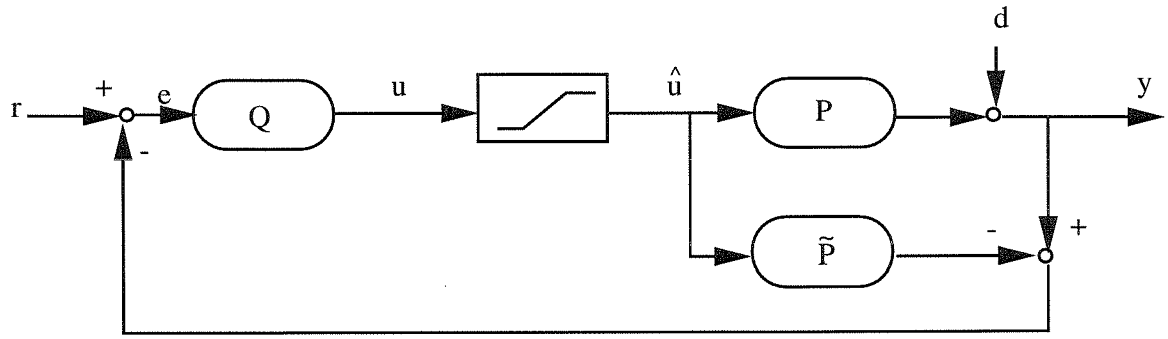

Figure A.1: IMC structure

tively. They are n by n transfer matrices. For $u \in \Re^n$, $\text{sat}(u) = \begin{cases} \text{sat}(u_1) \\ \vdots \\ \text{sat}(u_n) \end{cases}$, where

$\text{sat}(u_i) = \begin{cases} u_i^{max} & u_i > u_i^{max} \\ u_i & u_i^{min} \leq u_i \leq u_i^{max} \\ u_i^{min} & u_i < u_i^{min} \end{cases}$ denotes the input saturation function. For $x \in \Re^n$, $|x(t)|_1 = \sum_{i=1}^n |x_i(t)|$ denotes the 1-norm.

A.2 Problem Formulation

Consider the IMC structure as shown in Figure A.1. Define

$$y'(t) = (P * \hat{u})(t) + d(t) = \int_0^t P(t-\tau)\hat{u}(\tau)d\tau + d(t) \tag{A.1}$$

Thus y' corresponds to the output of the constrained system. Because of the saturation constraints, $y'(t)$ necessarily differs from $y(t)$, the output for the unconstrained system. In general, we would like to keep y' as close to y as possible. Mathematically, we want to solve the following optimization problem instantaneously at each time t.

$$\min_{\hat{u}} |(f * y)(t) - (f * y')(t)|_1 = \min_{\hat{u}} |(fPQ * e)(t) - (fP * \hat{u})(t)|_1 \tag{A.2}$$

where f is a filter such that fP is biproper. If P is strictly proper, then $\hat{u}$ does not affect y' instantaneously and the minimization is meaningless. Since our ultimate goal is to minimize $|y(t) - y'(t)|_1$, f must be diagonal in order not to introduce any change in the output direction.

The minimization is carried out continuously for $t \geq 0$. It is important to realize that this instantaneous minimization differs from the minimization over a horizon. For the conventional IMC structure displayed in Figure A.1, $\hat{u}(t) = \text{sat}(u(t)) = \text{sat}(\int_0^t Qe(\tau)d\tau)$ is completely determined for any given $e(t)$. Thus, in general, the conventional IMC implementation does not solve optimization problem (A.2) which optimizes the performance for the constrained system. In the next section, we will show that a modified IMC structure actually solves the optimization problem (A.2) instantaneously.

A.3 Anti-windup Design

We propose a modified IMC structure and show that it solves the optimization problem (A.2) instantaneously. The results are extended to the classical feedback structure. Several anti-windup algorithms are shown to be special cases. Furthermore, from our problem formulation, we can see what these methods do and what the consequences are.

A.3.1 IMC Structure

Figure A.2 shows the modified IMC structure where $Q = (I + Q_2)^{-1}Q_1$. Assume that Q is biproper.[1] We have

$$u(s) = Q_1 e(s) - Q_2 \hat{u}(s) = Q_1 e(s) - (Q_1 Q^{-1} - I)\hat{u}(s) \quad {}^{2} \qquad \text{(A.3)}$$

[1] Q is biproper if both Q and Q^{-1} are proper.

[2] Here zero initial condition is assumed. This is without loss of generality since Q is stable and nonzero initial conditions can be incorporated into $e(t)$.

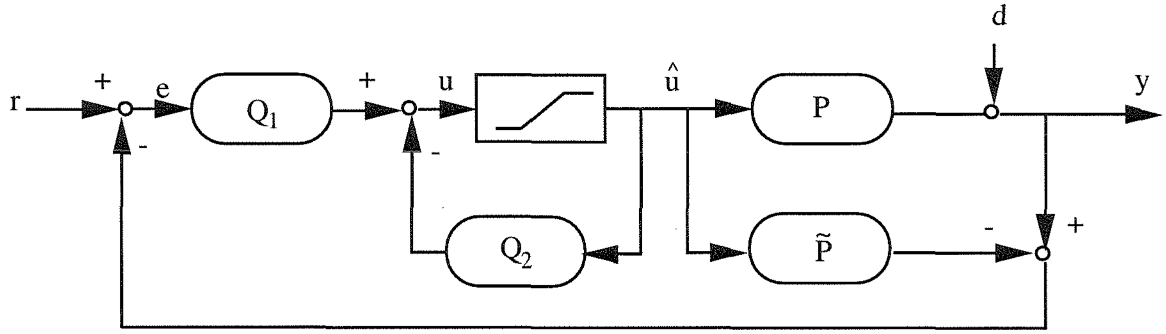

Figure A.2: Modified IMC structure

In the time domain,

$$u(t) - \hat{u}(t) = (Q_1 * e)(t) - (Q_1 Q^{-1} * \hat{u})(t) \tag{A.4}$$

The following lemma states how f should be chosen such that the modified IMC structure shown in Figure A.2 solves the optimization problem (A.2).

Lemma 16 *Suppose that Q is biproper and that $P = \tilde{P}$. If $fP|_{s=\infty}$ is a diagonal nonsingular matrix with finite elements and $Q_1 = fPQ$, then $\hat{u}(t)$ resulting from the modified IMC implementation (Figure A.2) is the solution of optimization problem (A.2). Furthermore, if $g = Df$ where D is a diagonal constant matrix, then the closed-loop responses with f and g are identical.*

Proof: $Q_1 = fPQ \Rightarrow u(t) - \hat{u}(t) = (fPQ*e)(t) - (fP*\hat{u})(t) = (f*y)(t) - (f*y')(t) \equiv y_f(t) - y'_f(t)$. We have

$$u_i(t) - \hat{u}_i(t) = y_{f_i}(t) - y'_{f_i}(t), \quad i = 1, 2, \cdots, n. \tag{A.5}$$

Since $fP|_{s=\infty}$ is diagonal, $\hat{u}_j, j \neq i$, do not affect y'_{f_i} instantaneously. Equations (A.5) can be solved independently for each $\hat{u}_i(t)$. Consider the first input, *i.e.* $i = 1$. When no saturation occurs at $t = t_1$, $\hat{u}_1(t_1) = u_1(t_1) = \mathrm{sat}(u_1(t_1))$ and $|y_{f_1}(t_1) - y'_{f_1}(t_1)| = 0$ is minimized. Suppose that saturation occurs at $t = t_2$, *i.e.* $u_1(t_2) > u_1^{max}$ or $u_1(t_2) < u_1^{min}$, we want to show that $\hat{u}_1(t_2) = \mathrm{sat}(u_1(t_2))$ also minimizes $|y_{f_1}(t_2) - y'_{f_1}(t_2)|$. Since $\hat{u}_1(t_2)$ affects $y'_{f_1}(t_2)$ linearly and $\hat{u}_j(t_2), j = 2, 3, \cdots, n$, do not affect $y'_{f_1}(t_2)$,

$|y_{f_1}(t_2) - y'_{f_1}(t_2)|$ is a convex function of $\hat{u}_1(t_2)$ only. If $\hat{u}_1(t_2) = u_1(t_2)$ for which $|y_{f_1}(t_2) - y'_{f_1}(t_2)| = 0$ is not feasible, *i.e.* $u_1(t_2) > u_1^{max}$

or $u_1(t_2) < u_1^{min}$, then the optimal solution which minimizes $|y_{f_1}(t_2) - y'_{f_1}(t_2)|$ must occur at the boundary, *i.e.* $\hat{u}_1(t_2) = \text{sat}(u_1(t_2))$. Therefore, choosing $\hat{u}_1(t) = \text{sat}(u_1(t))$ minimizes $|y_{f_1}(t) - y'_{f_1}(t)|$ for each $t \geq 0$. Since $|y_{f_i}(t) - y'_{f_i}(t)|$ is minimized for each i, $|y_f(t) - y'_f(t)|_1$ is minimized.

If $g = Df$, Equations (A.5) become

$$u_i(t) - \hat{u}_i(t) = D_{ii}[y_{f_i}(t) - y'_{f_i}(t)], \quad i = 1, 2, \cdots, n. \tag{A.6}$$

where $D = \text{diag}\{D_{11}, \cdots, D_{nn}\}$. Before saturation occurs, the system is unconstrained and $\hat{u}(t) = u(t)$ does not depend on D. Assume that system saturates for input 1 at $t = t_1$, then $\hat{u}_1(t_1) = u_1^{max}$ or $\hat{u}_1(t_1) = u_1^{min}$. As long as the right hand side of Equation (5) does not become zero for $i = 1$, input 1 stays saturated and $\hat{u}_1(t)$ is constant during this period. Input 1 becomes unsaturated only if the right hand side of Equation (5) becomes zero for $i = 1$ which is not a function of D_{11}. Therefore, the system comes out of the saturation at the same time regardless of what D_{11} is. Similar arguments can be used when more than one input saturates. Therefore, the closed-loop responses for f and g are identical. □

Remark 50 *If $fP|_{s=\infty}$ is not diagonal, then $y'_{f_i}(t)$ may also be affected by $\hat{u}_j(t), j \neq i$, instantaneously. The convexity argument would not work since $|y_{f_i}(t) - y'_{f_i}(t)|$ is also affected by $\hat{u}_j(t), j \neq i$.*

Remark 51 *f must be diagonal in order not to introduce any change in the output direction. However, f for which $fP|_{s=\infty}$ is diagonal may not be diagonal. To get around this problem, we can design a diagonal f for $\tilde{P}$ such that $f\tilde{P}|_{s=\infty}$ is diagonal. $\tilde{P}$ can be chosen arbitrarily close to P. Q_2 must be strictly proper to be implementable. This can be achieved by choosing f appropriately.*

Remark 52 *Q is usually minimum phase and always stable. If Q is minimum phase and Q_1 non-minimum phase, then $(I + Q_2)^{-1}$ must be unstable. Therefore, Q_1 must* *be minimum phase and stable to guarantee internal stability of the closed-loop system. f must be chosen such that fPQ is both minimum phase and stable.*

Remark 53 *For the modified IMC structure, the input is kept saturated for an optimal amount of time until $|y_f(t) - y'_f(t)|$ becomes zero. Thus, in general, the performance is greatly improved when f is appropriately chosen.*

Different controller factorizations can be obtained by choosing f differently. We discuss two special cases here.

Case 1: $f = P^{-1}$. The optimization problem (A.2) becomes $\min_{\hat{u}} |u(t) - \hat{u}(t)|_1$. The solution corresponds to the conventional IMC structure which "chops off" the control input resulting in performance deterioration. However, stability of the closed-loop system is guaranteed.

Case 2: f is such that Q_1 is a constant matrix. The optimization becomes $\min_{\hat{u}} |Q_1[e(t) - e'(t)]|_1$, where $e'(t) = (Q^{-1} * \hat{u})(t)$. This factorization corresponds to the Model State Feedback proposed by Coulibaly et al. [17] for SISO systems. The same factorization has also been proposed recently by Goodwin et al. [34] where Q_1 is chosen to be $Q(\infty)$. Thus, these are special cases of the factorization we present.

The performance in this case is greatly improved, but stability of the closed-loop system is not guaranteed. If the dynamics of PQ are slow, however, minimizing the weighted controller input error $(e(t) - e'(t))$ may not be a good way to optimize the nonlinear performance. After the system comes out of the nonlinear region, the controller takes no action to compensate for the effect of the error, $e(t) - e'(t)$, introduced during the saturation.

In Case 1 f was chosen to guarantee stability while f was chosen to enhance performance in Case 2. Therefore, f can generally be tuned to trade off performance and stability of the constrained system. It should be pointed out that f in Case 2 was not an extreme choice.

A.3.2 Classical Feedback Structure

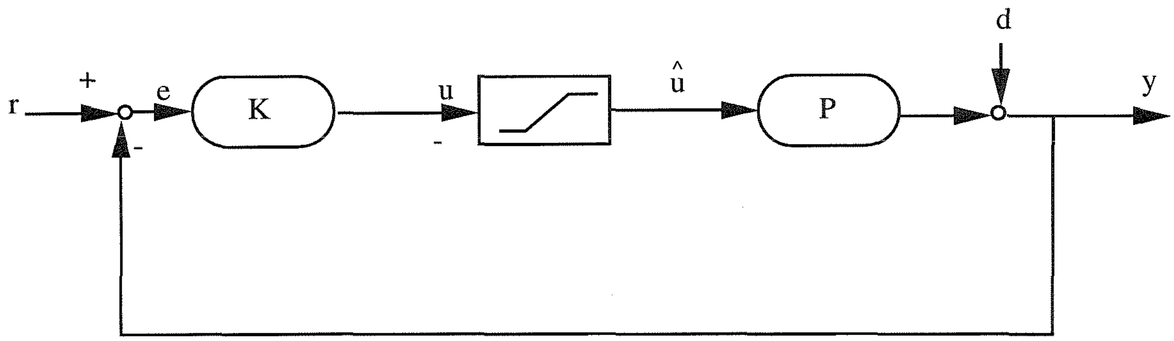

Figure A.3: Classical feedback structure

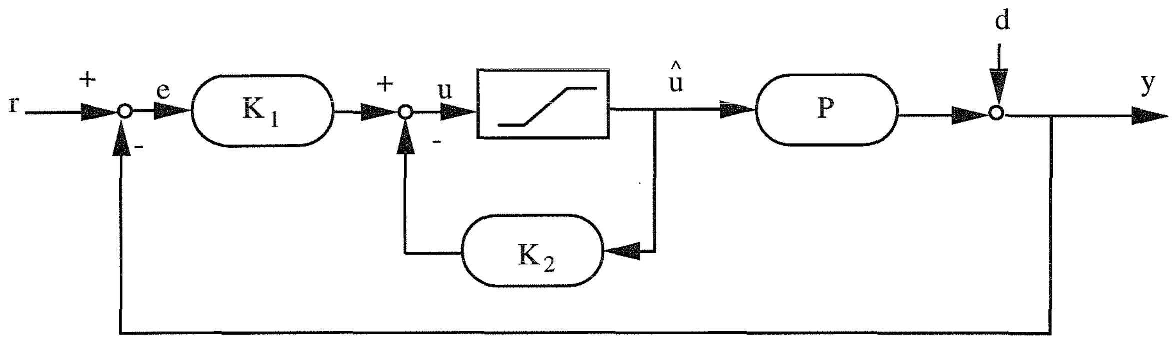

Figure A.4: Classical feedback structure with anti-windup

For stable systems, the IMC structure shown in Figure A.1 and the classical feedback structure shown in Figure A.3 are equivalent. The results for the modified IMC structure can be extended directly to the classical feedback structure to obtain the anti-windup structure shown in Figure A.4. The controllers K_1 and K_2 are defined as follows:

$$K_1 = Q_1 \tag{A.7}$$

$$K_2(s) = Q_2 - Q_1\tilde{P} \tag{A.8}$$

Hanus et al. [39, 40] suggested the following

$$K_1 = K(\infty) \tag{A.9}$$

$$K_2(s) = K_1 K^{-1}(s) - I \tag{A.10}$$

where $K = Q(I - \tilde{P}Q)^{-1}$. This factorization corresponds to $f = K_1 Q^{-1} P^{-1}$. Therefore, Hanus' conditioning technique minimizes $|K_1[e(t) - e'(t)]|_1$. In general, f chosen in a such way is *not* diagonal in general. While this does not matter for SISO systems, it introduces undesirable change in the output direction and results in poor performance (see Example 3) for MIMO systems.

A.4 Examples

In this section, several examples are shown to demonstrate the effectiveness of the proposed method.

Example 11 Consider the following plant:

$$P(s) = \frac{2}{100s + 1} \tag{A.11}$$

The IMC controller designed for a step input is

$$Q(s) = \frac{100s + 1}{2(20s + 1)} \tag{A.12}$$

Case 1. Choosing $f = 2.5(20s + 1)$ [3] gives

$$\begin{aligned} Q_1 &= 2.5 \\ Q_2 &= \frac{4}{100s + 1} \end{aligned}$$

Case 2. Choosing $f = 50(s + 1)$ gives

$$\begin{aligned} Q_1 &= \frac{50(s + 1)}{20s + 1} \\ Q_2 &= \frac{99}{100s + 1} \end{aligned}$$

[3] The constant 2.5 is such that Q_2 is strictly proper.

The input is constrained between the saturation limits ± 1. The responses to a unit step disturbance with the conventional IMC and the modified IMC implementations are shown in Figures A.5 and A.6 along with the unconstrained responses. The figures illustrate the sluggishness of performance of the conventional IMC implementation when the closed loop dynamics are much faster than those of the open loop. For the conventional IMC implementation, the saturation effectively "chops off" the control input resulting in performance deterioration. The modified IMC implementation keeps the control signal saturated for an optimum length of time as discussed in Section 3 resulting in improved performance. f in Case 1 corresponds to minimizing $|e(t) - e'(t)|$ while f in Case 2 corresponds approximately to minimizing $|y(t) - y'(t)|$. The control input in Case 2 stays saturated until $y(t) \approx y'(t)$ while the control input in Case 1 stays saturated until $e(t) = e'(t)$. In Case 1, the difference between $y(t)$ and $y'(t)$ resulting from the difference between $e(t)$ and $e'(t)$ during the saturation is not compensated as can be seen in Figure A.5.

Example 12 This example is taken from Doyle et al. [26] where the conventional anti-windup method did not result in a stable closed loop system. The plant is a fourth order lag-lead butterworth:

$$P = 0.2\left(\frac{s^2 + 2\xi_1\omega_1 s + \omega_1^2}{s^2 + 2\xi_1\omega_2 s + \omega_2^2}\right)\left(\frac{s^2 + 2\xi_2\omega_1 s + \omega_1^2}{s^2 + 2\xi_2\omega_2 s + \omega_2^2}\right) \tag{A.13}$$

where $\omega_1 = 0.2115, \omega_2 = 0.0473, \xi_1 = 0.3827$ and $\xi_2 = 0.9239$.

The IMC controller is

$$Q = \frac{s+1}{(16s+1)P} \tag{A.14}$$

Choosing $f = \frac{5(16s+1)}{16(s+1)}$ gives

$$\begin{aligned} Q_1 &= \frac{5}{16} \\ Q_2(s) &= \frac{5}{16Q(s)} - 1 \end{aligned}$$

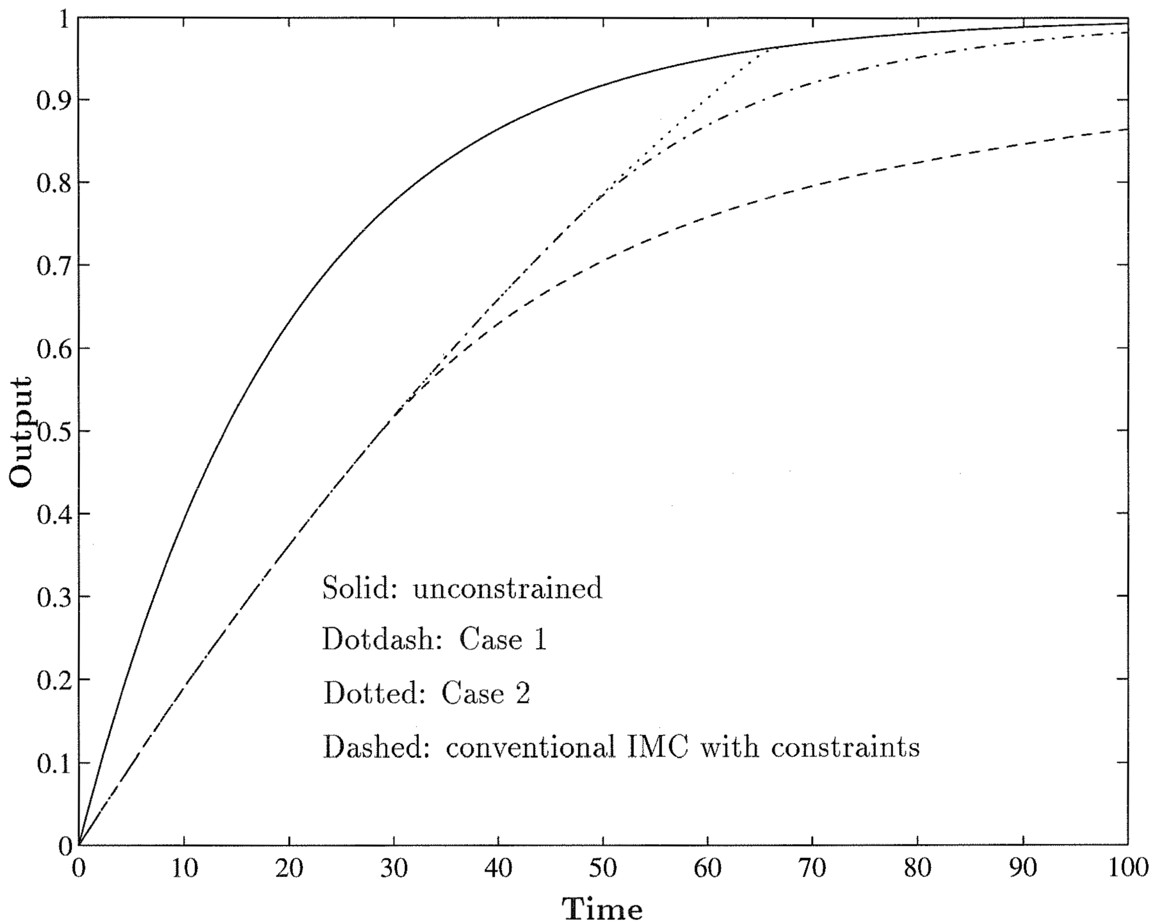

Figure A.5: Example 11—Plant output responses

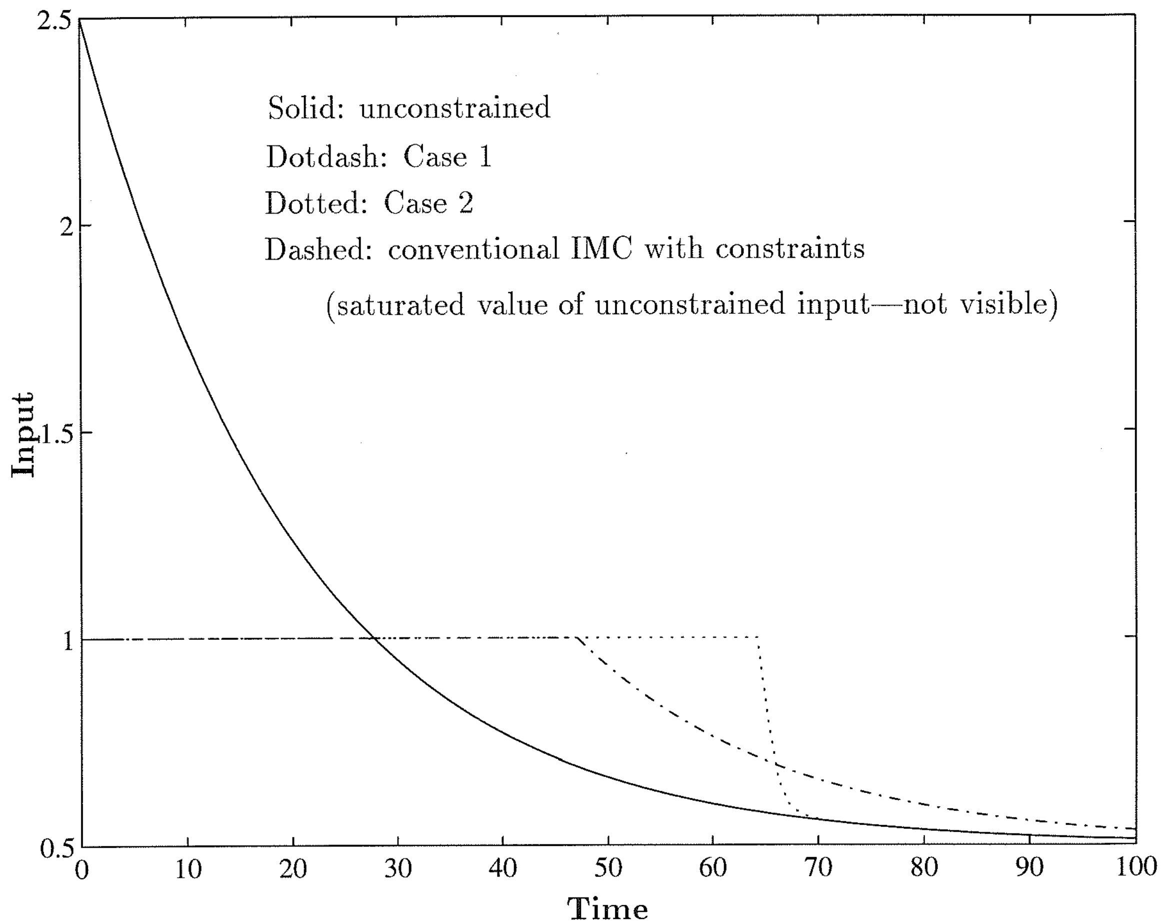

Figure A.6: Example 11—Controller output responses

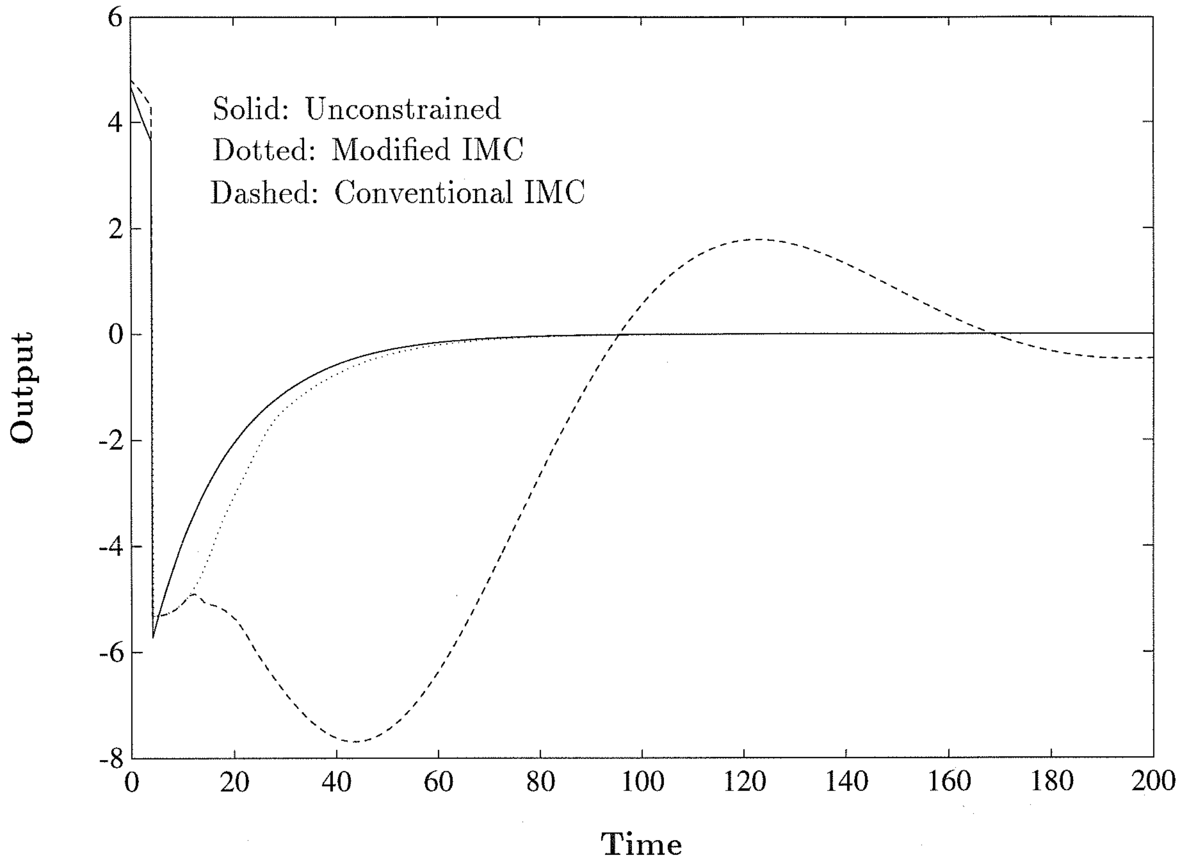

Figure A.7: Example 12—Plant output responses

The input is constrained between the saturation limits ± 1. Figure A.7 shows the responses for a disturbance input with step of magnitude of 5 at time $t = 0$ and a switch to -5 at $t = 4$. The performance improvement over the conventional IMC implementation is significant. Furthermore, the off-axis criterion [12] can be used to show that the closed-loop system is globally asymptotically stable.

Example 13 Consider the following plant:

$$P(s) = \frac{10}{100s+1} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \tag{A.15}$$

Both inputs are constrained between the saturation limits ± 1. A setpoint change of $[0.63\ 0.79]^T$ is applied. The IMC controller designed for a step input is

$$Q(s) = \frac{100s+1}{10(20s+1)} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \tag{A.16}$$

Two values of f , one diagonal and one non-diagonal, are chosen to see how f (diagonal or not diagonal) affects the closed-loop performance.

Case 1.

$$\begin{aligned} f &= 10(s+1) \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \\ Q_1 &= fPQ \\ Q_2 &= fP - I \end{aligned}$$

Case 2.

$$\begin{aligned} f &= 2.5(s+1)I \\ \tilde{P} &= \frac{10}{100s+1} \begin{bmatrix} 4 & \frac{-5}{0.1s+1} \\ \frac{-3}{0.1s+1} & 4 \end{bmatrix} \\ Q_1 &= f\tilde{P}Q \\ Q_2 &= f\tilde{P} - I \end{aligned}$$

The responses for both cases and the conventional IMC implementation are shown in Figure A.7. As we can see, choosing f to be a diagonal nonsingular matrix is crucial to obtain good nonlinear performance.

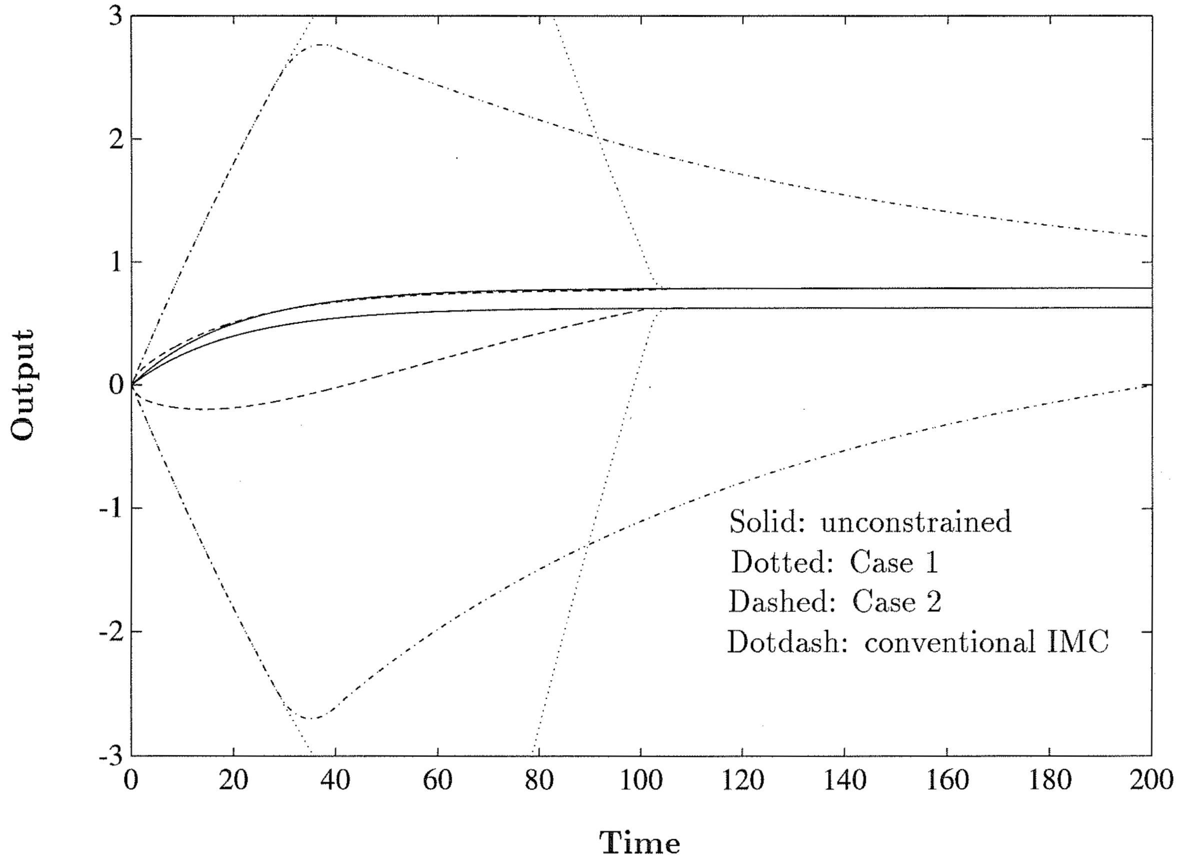

Figure A.8: Example 13—Plant output responses

A.5 Conclusions

We have proposed an anti-windup scheme which optimizes the error between the constrained and the unconstrained outputs of the system. The method generalizes the Model State Feedback for SISO systems proposed in Coulibaly et al. [17] and Hanus's conditioning technique. In particular, the Model State Feedback corresponds to choosing f such that Q_1 is constant; Hanus's conditioning technique corresponds to choosing f such that $Q_1 = K(\infty)$; the factorization proposed by Goodwin et al. [34] corresponds to choosing f such that $Q_1 = \lim_{s \to \infty} Q(s)$. Furthermore, from our problem formulation, we can see what these methods do and what the consequences are. As shown by Example 3, the performance for $Q_1 = K(\infty)$ for MIMO systems may suffer when $K(\infty)$ is not diagonal. Examples illustrate that this scheme provides graceful degradation of performance.

The attractive features of the scheme are its simplicity and effectiveness. The filter f can be tuned to trade off performance and stability of the constrained system. However, a rigorous and nonconservative stability analysis needs to be developed. Recently, Campo [9] and Kothare et al. [48] unified all existing AWBT schemes and developed a general framework for studying stability and robustness issues. The importance of this work lies in that model uncertainty can be taken into account systematically and powerful theory exists to analyze the closed loop system for stability and robustness. However, their analysis is based on the standard conic sector nonlinear stability theory. Therefore, the results could be potentially conservative. Another drawback for all AWBT schemes is their inability to handle output constraints.

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